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A TRUSTWORTHY COMPUTATION OF THE TRANSITORY AND STATIONARY RESPONSES FOR THE SHIP ROLLING BY A MODIFIED MULTIPLE TIME SCALE METHOD

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Abstract: An important step in any analysis concerning the ship behaviour in waves is a reasonable accurate assessment of high roll amplitudes. The most commonly used way to estimate the roll oscillations is to solve a nonlinear differential equation with the roll angle as unknown. The nonlinearities refer to different ways of approximating the damping and restoring moments. The lack of exact analytical solutions and the inability of a numerical solution to highlight the influence of various parameters on the ship motion, leads to the need to use approximate analytical methods. Operating the benefits of a modified multiple time scale method recently proposed in the literature, we obtained the governing equations for the transition towards the stationary solutions, the first – order approximations for these solutions and the frequency – amplitude relationship. The accuracy of the analytical results derived in the paper was demonstrated by comparing them with the numerical results (considered to be exact). The agreement between the two was excellent for relatively small forcing amplitude and satisfactory for high level of them on almost the entire range of used values for excitation frequency, including the primary resonance domain.
K***ey words:*** ship rolling, amplitudes, accuracy, frecquency.

1. INTRODUCTION

Between the six motions of the ship, the roll motion is the most critical one because it can lead to the ship capsizing. If we presume small motions in all modes excepting roll and the existence of a coordinate system origin, these assumptions lead to an equation of motion in which roll is uncoupled from the other five degrees of freedom, [1]. This equation is linear for small angles of roll motions. For medium or severe oscillations, the linear theory ceases to accurately describe the ship's roll. Nonlinear contributions, especially associated with damping and restoring moments, and mathematically modelled in the form of higher degree terms of some polynomials, need to be added to the equation of motion.

A significant number of nonlinear models of the decoupled roll equation have been proposed by researchers in the field. Thus, Dalzell [2] and Cardo et all [3] considered a simple expression for the damping moment with only linear, quadratic and/or cubic terms. A good approximation of the restoring moment can be achieved by means of small degree polynomials (usually three) in the roll angle as a variable, [4]. As for the hydrodynamic moment, it can be introduced in the equation as a truncated series expansion about the ship speed, [5].

Even if there is a natural temptation to introduce into the equation of motion a wide range of parameters that can have a certain effect on the behaviour of the ship, this would only make impossible an exact analytical solution. A numerical solution, obtained with a quality software package such as Matlab, could be a substitute but the price paid is an incomplete identification of the influence of different terms of the equation. To get out of this vicious circle and get a starting picture of the ship's dynamics, one can start from simplified expressions of hydrodynamic, damping and restoring moments and use approximate analytical approaches to allow an estimate of the transient and stationary phases of roll oscillations, [6].

Even for this simplified form of the roll equation, in the version in which the study of large amplitude oscillations and, possibly, of the ship capsizing is desired, the traditional approximate techniques (eg. the method of harmonic balance, the Lindstedt – Poincare method etc.) provide solutions with a low degree of accuracy, [7 - 9].

Four years ago, Razzak et al. proposed a modified multiple time scale approach to solve strongly nonlinear equations and applied it to the forced damped Duffing and Van der Pol oscillators [10]. In the paper, we considered their version to determine the transitory and stationary solutions of a nonlinear roll equation and compared the obtained solutions with their numerical counterparts.

2. THE MODIFIED MULTIPLE TIME SCALE METHOD

It will take into consideration the nonlinear differential equation of second order presented in equation (1)

$$\ddot{x} + 2k\dot{x} + \omega_0^2 x = \varepsilon f(x, \dot{x}) + \varepsilon A\, cos\nu t \quad (1)$$

where k, ω_0 and ν are positive constants, f is a nonlinear function such that $f(-x, -\dot{x}) = -f(x, \dot{x})$, A is the amplitude of the external excitation and ν its frequency. The parameter ε could be either small or large and dots indicate the order of differentiation with respect to time variable *t*.
The steps of the method are as follows:

*i)*Choose a first approximate solution in the form

$$x(t, \varepsilon) = a_1(t) + a_2(t) + \varepsilon x_1(a_1, a_2) + \cdots \quad (2)$$

The variables a_1 and a_2 depend on times $t_0, t_1, t_2, ...$ where $t = t_0 + \varepsilon t_1 + \varepsilon^2 t_2 + \cdots$

ii) Rewrite the equation (1) as

$$(D - \lambda_1)(D - \lambda_2)x(t) = \varepsilon f(x(t), \dot{x}(t)) + \varepsilon A\, cos\nu t \quad (3)$$

where $\lambda_{1,2} = -k \pm i\omega, \omega = \sqrt{\omega_0^2 - k^2}, k < \omega,$ are the roots of the characteristic equation $\lambda^2 + 2k\lambda + \omega_0^2 = 0$ associated to the unperturbed case ($\varepsilon = 0$) and $D = D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \cdots, D_k = d/dt_k, k = 0, 1, 2, .., \; D_0 a_j = \lambda_j a_j, j = 1, 2$ (4)

iii) Replace the assumed solution (2) into (3) and equate the coefficients of ε

$$(D_0 - \lambda_2)(D_1 a_1) + (D_0 - \lambda_1)(D_1 a_2) + (D_0 - \lambda_1)(D_0 - \lambda_2)x_1 = f + A\, cos\nu t \quad (5)$$

iv) Expand the nonlinear function *f* in a Fourier series

$$f = \sum_{m_1, m_2 = 0}^{\infty} F_{m_1, m_2}\, e^{(m_1\lambda_1 + m_2\lambda_2)t} \quad (6)$$

v) Separate equation (5) into three parts for $D_1 a_1, D_1 a_2$ and x_1 as

$$\begin{cases} (D_0 - \lambda_2)(D_1 a_1) = \sum_{m_1, m_2 = 0}^{\infty} F_{m_1, m_2}\, e^{(m_1\lambda_1 + m_2\lambda_2)t} + \frac{A e^{i\nu t_0}}{2}, m_1 - m_2 = 1 \\ (D_0 - \lambda_1)(D_1 a_2) = \sum_{m_1, m_2 = 0}^{\infty} F_{m_1, m_2}\, e^{(m_1\lambda_1 + m_2\lambda_2)t} + \frac{A e^{-i\nu t_0}}{2}, m_1 - m_2 = -1 \\ (D_0 - \lambda_1)(D_0 - \lambda_2)x_1 = \sum_{m_1, m_2 = 0}^{\infty} F_{m_1, m_2}\, e^{(m_1\lambda_1 + m_2\lambda_2)t}, m_1 - m_2 \neq \pm 1 \end{cases} \quad (7)$$

This separation has the effect of not including secular terms of type $t\, sint$ or $t\, cost$ in the component x_1 (that is, making it unbounded) [11].

vi) Use the polar coordinates

$$a_{1,2} = a e^{\pm i\varphi}/2 \quad (8)$$

to transform the first two equations of system (7) into a system in amplitude *a* and phase φ. This system describes both the transient and stationary processes.

vii) Determine the first – order component x_1 from the third equation (7) with *a* and φ determined in the previous step.

3. THE NONLINEAR ROLL EQUATION

The nonlinear roll equations suggested by researchers differ mainly in the way in which the damping and restoring moments are represented. The equation used in this paper for describing the roll motion of a ship in regular waves is written as

$$\ddot{\theta} + d_1\dot{\theta} + d_3\dot{\theta}^3 + k_1\theta + k_3\theta^3 = m\, cos\Omega t \quad (9)$$

with d_1 and d_3 denoting the linear and cubic roll damping coefficients, $k_1$and k_3 the coefficients of the roll restoring moment (written as a third order odd polynomial), *m* and Ω the forcing amplitude and the wave frequency, respectively. Additionally, θ represents the roll angle and a dot denotes the time differentiation.
For the estimation of ship behaviour, the coefficients $d_i, i = 1,3$ and $k_i, i = 1,3$ were adopted for an existing ferry ship [12]:

$$d_1 = 0.01265913\ s^{-1}, d_3 = 0.4954\ s,$$
$$k_1 = 0.691997033\ s^{-2}, k_3 = -0.53920393\ s^{-2}$$

4. APPROXIMATE ANALYTICAL SOLUTION OF THE ROLL EQUATION

To apply the algorithm described above, one use the notations $k_3 = -\varepsilon\tilde{k}_3, d_3 = -\varepsilon\tilde{d}_3$ and $m = \varepsilon\tilde{m}$ and rewrite the equation (9) as

$$\ddot{\theta} + d_1\dot{\theta} + k_1\theta = \varepsilon\left(\tilde{k}_3\theta^3 + \tilde{d}_3\dot{\theta}^3\right) + \varepsilon\tilde{m}\, cos\Omega t \quad (10)$$

The characteristic equation $\lambda^2 + d_1\lambda + k_1 = 0$ has the roots $\lambda_{1,2} = \alpha \pm i\beta$, with $\alpha = -d_1/2$ and $\beta = \sqrt{4k_1 - d_1^2}/2$. The first – order approximate solution is chosen in the form

$$\theta(t, \varepsilon) = \theta_1(t) + \theta_2(t) + \varepsilon\Theta(\theta_1, \theta_2) \quad (11)$$

Noting that $D_0\theta_j = \lambda_j\theta_j, j = 1,2$, the nonlinear function $f(\theta,\dot{\theta}) = \tilde{k}_3\theta^3 + \tilde{d}_3\dot{\theta}^3$ can be expanded as

$$f = \tilde{k}_3(\theta_1+\theta_2+\varepsilon\Theta)^3 + \tilde{d}_3[(D_0+\varepsilon D_1)(\theta_1+\theta_2+\varepsilon\Theta)]^3 = \tilde{k}_3(\theta_1+\theta_2+\varepsilon\Theta)^3 + \tilde{d}_3\big(\lambda_1\theta_1+\lambda_2\theta_2+O(\varepsilon)\big)^3$$
$$= \big(\tilde{k}_3+\tilde{d}_3\lambda_1^3\big)\theta_1^3 + \big(\tilde{k}_3+\tilde{d}_3\lambda_2^3\big)\theta_2^3 + 3\big(\tilde{k}_3+\tilde{d}_3\lambda_1^2\lambda_2\big)\theta_1^2\theta_2 + 3\big(\tilde{k}_3+\tilde{d}_3\lambda_1\lambda_2^2\big)\theta_1\theta_2^2$$

Applying the separation rule (7), one obtains the following equations for $D_1\theta_1$, $D_1\theta_2$ and Θ:

$$\begin{cases}(D_0-\lambda_2)(D_1\theta_1) = 3\big(\tilde{k}_3+\tilde{d}_3\lambda_1^2\lambda_2\big)\theta_1^2\theta_2 + \tilde{m}e^{i\Omega t_0}/2\\ (D_0-\lambda_1)(D_1\theta_2) = 3\big(\tilde{k}_3+\tilde{d}_3\lambda_1\lambda_2^2\big)\theta_1\theta_2^2 + \tilde{m}e^{-i\Omega t_0}/2\\ (D_0-\lambda_1)(D_0-\lambda_2)\Theta = \big(\tilde{k}_3+\tilde{d}_3\lambda_1^3\big)\theta_1^3 + \big(\tilde{k}_3+\tilde{d}_3\lambda_2^3\big)\theta_2^3\end{cases} \quad (12)$$

Assuming a particular solution of form $D_1\theta_1 = l_1\theta_1^2\theta_2$, $l_1$constant, for the first equation (12) without the external term $\tilde{m}e^{i\omega t_0}/2$, and noting that

$$(D_0-\lambda_2)(l_1\theta_1^2\theta_2) = D_0(l_1\theta_1^2\theta_2) - \lambda_2 l_1\theta_1^2\theta_2 = 2l_1\lambda_1\theta_1^2\theta_2 = 3\big(\tilde{k}_3+\tilde{d}_3\lambda_1^2\lambda_2\big)\theta_1^2\theta_2$$

one gets $l_1 = \frac{3(\tilde{k}_3+\tilde{d}_3\lambda_1^2\lambda_2)}{2\lambda_1}$. Adding the particular solution associated to external term, $D_1\theta_1 = \frac{\tilde{m}}{2(i\omega-\lambda_2)}e^{i\Omega t_0}$, one has

$$D_1\theta_1 = \frac{3\big(\tilde{k}_3+\tilde{d}_3\lambda_1^2\lambda_2\big)}{2\lambda_1}\theta_1^2\theta_2 + \frac{\tilde{m}}{2(i\omega-\lambda_2)}e^{i\Omega t_0}$$

But

$$\dot{\theta}_1 = D\theta_1 = (D_0+\varepsilon D_1)\theta_1 = \lambda_1\theta_1 + \varepsilon D_1\theta_1 =$$
$$= (\alpha+i\beta)\theta_1 + \varepsilon\frac{3\big[\tilde{k}_3+\tilde{d}_3(\alpha+i\beta)^2(\alpha-i\beta)\big]}{2(\alpha+i\beta)}\theta_1^2\theta_2 + \frac{\varepsilon\,\tilde{m}}{2(i\omega-\alpha+i\beta)}e^{i\Omega t_0}$$

Now, one consider the polar form of components θ_1 and θ_2

$$\theta_{1,2}(t) = \frac{1}{2}a(t)e^{\pm i\varphi(t)}$$

and calculate the derivative $\dot{\theta}_1 = \frac{1}{2}(\dot{a}+ia\dot{\varphi})e^{i\varphi}$. Equating the two representations of the derivative $\dot{\theta}_1$ and dissociating the real, respectively, imaginary parts, one obtains (after some algebra) the following system in the amplitude a and phase φ:

$$\begin{cases}\dot{a} = \alpha a - \varepsilon\tilde{m}\frac{\alpha\cos\psi+(\Omega+\beta)\sin\psi}{\alpha^2+(\Omega+\beta)^2} + \frac{3\varepsilon a^3}{8}\frac{\tilde{k}_3\alpha+\tilde{d}_3(\alpha^2+\beta^2)^2}{\alpha^2+\beta^2}\\ a\dot{\varphi} = \beta a - \varepsilon\tilde{m}\frac{(\Omega+\beta)\cos\psi-\alpha\sin\psi}{\alpha^2+(\Omega+\beta)^2} - \frac{3\varepsilon a^3}{8}\frac{\tilde{k}_3\beta}{\alpha^2+\beta^2}\end{cases}$$

with $\psi = \varphi - \Omega t_0$. This system describes the transition towards the steady – state solution.

The next step of the procedure is to find the component Θ from the third equation (12). One may search for a solution of form $\Theta = C_1\theta_1^3 + C_2\theta_2^3$. One can immediately check that

$$(D_0-\lambda_2)(C_1\theta_1^3) = C_1(3\lambda_1-\lambda_2)\theta_1^3,\ (D_0-\lambda_1)(D_0-\lambda_2)(C_1\theta_1^3) = 2C_1\lambda_1(3\lambda_1-\lambda_2)\theta_1^3$$

so that $C_1 = \frac{\tilde{k}_3+\tilde{d}_3\lambda_1^3}{2\lambda_1(3\lambda_1-\lambda_2)}$. Similarly, $C_2 = \frac{\tilde{k}_3+\tilde{d}_3\lambda_2^3}{2\lambda_2(3\lambda_2-\lambda_1)}$. Using the polar coordinates, one deduce that

$$\Theta = \left[\frac{\tilde{k}_3+\tilde{d}_3\lambda_1^3}{2\lambda_1(3\lambda_1-\lambda_2)}e^{3i\varphi} + \frac{\tilde{k}_3+\tilde{d}_3\lambda_2^3}{2\lambda_2(3\lambda_2-\lambda_1)}e^{-3i\varphi}\right]a^3 = \big(\tilde{S}_1\cos3\varphi + \tilde{S}_2\sin3\varphi\big)a^3$$

where

$$\tilde{S}_1 = \frac{\tilde{k}_3+\tilde{d}_3\lambda_1^3}{16\lambda_1(3\lambda_1-\lambda_2)} + \frac{\tilde{k}_3+\tilde{d}_3\lambda_2^3}{16\lambda_2(3\lambda_2-\lambda_1)} = \frac{(\alpha^2-2\beta^2)\tilde{k}_3 + (\alpha^2+\beta^2)(\alpha^3+3\alpha\beta^2)\tilde{d}_3}{16(\alpha^2+\beta^2)(\alpha^2+4\beta^2)}$$

$$\tilde{S}_2 = \frac{\tilde{k}_3+\tilde{d}_3\lambda_1^3}{16\lambda_1(3\lambda_1-\lambda_2)} - \frac{\tilde{k}_3+\tilde{d}_3\lambda_2^3}{16\lambda_2(3\lambda_2-\lambda_1)} = \frac{3\alpha\beta\tilde{k}_3 - 2(\alpha^2+\beta^2)\beta^3\tilde{d}_3}{16(\alpha^2+\beta^2)(\alpha^2+4\beta^2)}$$

Returning to the initial parameters k_3, d_3 and m, using the angle ψ instead of φ, we obtain for the first order approximate solution of roll equation (9) the expression

$$\theta_{apx}(t) = a(t)\cos\big(\Omega t + \psi(t)\big) + \varepsilon\Theta(t) \quad (13)$$

where $a(t)$ and $\psi(t)$ are the solutions of the system

$$\begin{cases}\dot{a} = \alpha a - m\frac{\alpha\cos\psi+(\Omega+\beta)\sin\psi}{\alpha^2+(\Omega+\beta)^2} - \frac{3a^3}{8}\frac{k_3\alpha+d_3(\alpha^2+\beta^2)^2}{\alpha^2+\beta^2}\\ \dot{\psi} = \beta - \Omega - \frac{m}{a}\frac{(\Omega+\beta)\cos\psi-\alpha\sin\psi}{\alpha^2+(\Omega+\beta)^2} + \frac{3a^2}{8}\frac{k_3\beta}{\alpha^2+\beta^2}\end{cases}$$

and

$$\varepsilon\Theta = -[S_1\cos3(\Omega t+\psi) + S_2\sin3(\Omega t+\psi)]a^3$$

S_1 and S_2 are obtained from $\tilde{S}_1$ and $\tilde{S}_2$ based on substitutions $(\tilde{k}_3,\tilde{d}_3)\to(k_3,d_3)$. Finally, the expression (13) approximates the exact solution both in the starting phase of the ship roll oscillation and in the steady - state one.

5. NUMERICAL SIMULATIONS

To examine the modified multiple scale method's performance, in this section we compared the analytical solution (13) with its numerical counterpart. All computations and plots were made with Matlab softwere package. In all the graphical representations, the international system of units was used.

Cumulative data over the past few decades as well as numerical and experimental simulations of the capsizing phenomenon show that it occurs as a result

of a relatively short sequence of steep waves acting on the vessel from the side. The conditions $\theta > 1.5\ rad$ and $t < 10\ T$ were used to divide the parameter plane (Ω, m) into two distinct regions. Thus, the dark area in figure 1 is associated to those pairs (Ω, m) for which equation (9) has bounded numerical solutions while the light region reveals a capsized ship. Because we were interested in comparing finite solutions, all the pairs (Ω, m) used in the research corresponded to the dark area.

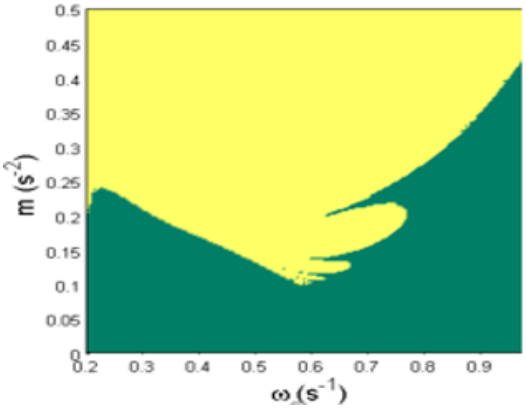
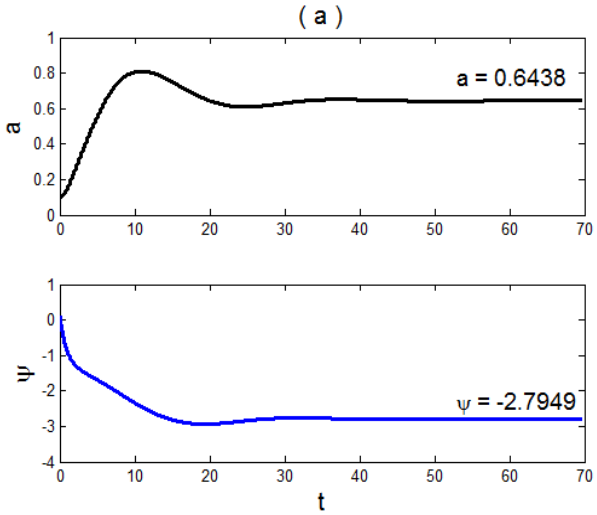


Fig. 1. The light (yellow) colour in the parameter plane (Ω, m) shows a capsized vessel while the dark (green) colour stands for an oscillating one.

The transition to the steady state solution may take relatively little time, in which case the graphs of the functions $a(t)$ and $\psi(t)$ resemble those associated with an overdamped oscillator (see figure 2*a*), or it takes hundreds of seconds to overcome the transition, as in the case of an underdamped oscillator (see figure 3*a*). After this period, the amplitude a and phase ψ remain constant.
Regardless of the type of transition, the first order approximation (13) is in good and even excellent agreement with the exact solution, especially for the steady state of the oscillations (see figures 2*b* and 3*c*).



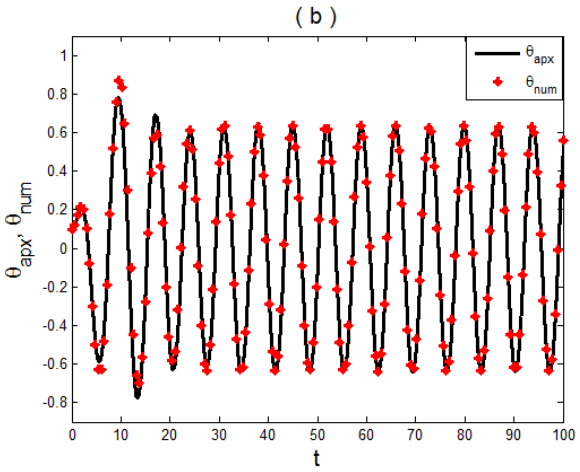
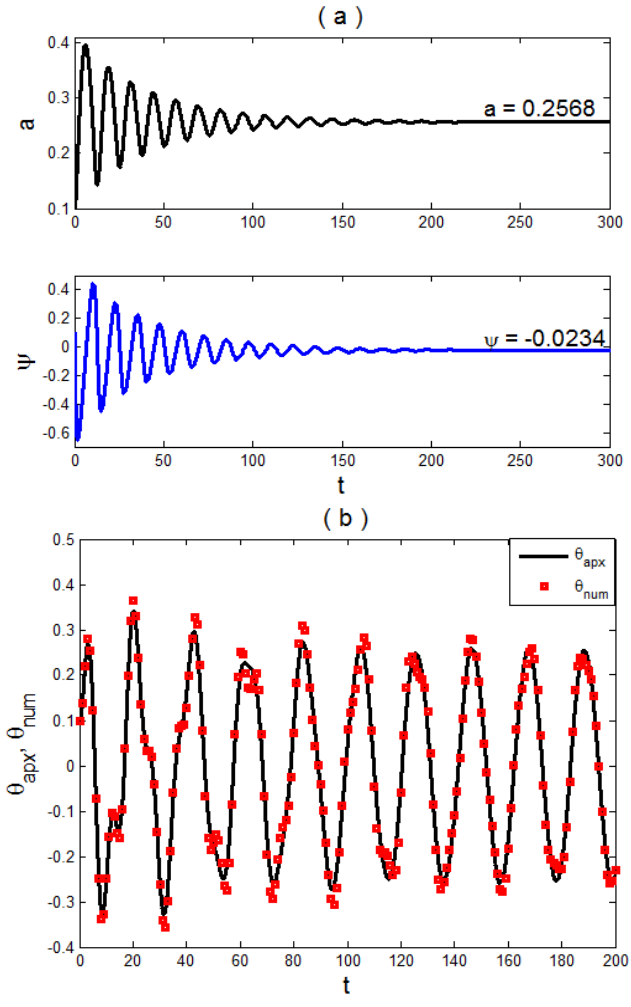


Fig. 2. (*a*) The time evolution of amplitude a and phase ψ for m = 0.2 and $\Omega = 0.9$; (*b*) Numerical solution versus approximation (13).

Generally, the term $\varepsilon\Theta$ has a minor contribution to the approximate solution, as illustrated in figure 3*d*.
A useful indicator of the ship's behaviour in waves is the so-called *frequency – response curve* that indicates the roll amplitude as a function of Ω for a fixed forcing amplitude m. To construct it, the equation (9) was integrated for $t \in [0, 50 \cdot \frac{2\pi}{\Omega}]$ (large enough for the transition period to end).



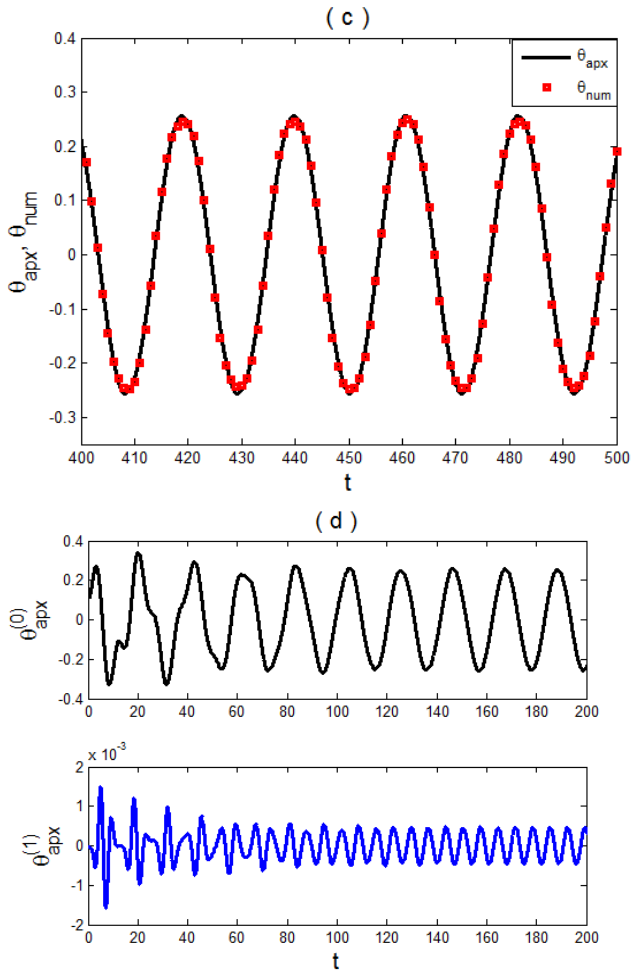


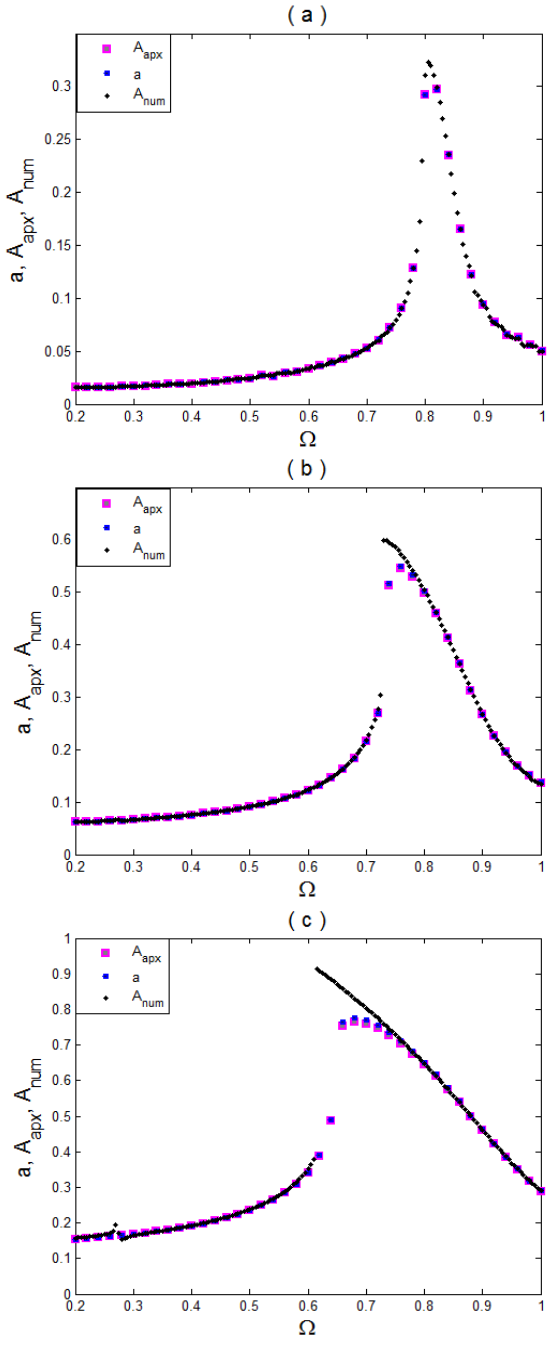
Fig. 3. (*a*) The time evolution of amplitude *a* and phase ψ for m = 0.15 and $\Omega = 0.3$;
(*b*) Numerical solution versus approximation (13) in the transition phase; (*c*) The same as in (*b*) but for the steady – state period; *(d)* Separation into components of the approximate solution (13).

The interval $\Omega \in [0.2, 1]$ was chosen for wave frequency. The average oscillation amplitude of the last five forcing periods (considered to be part of the steady state) was compared in figure 4 with its counterpart produced by the approximate expression (13). Numerical results are labelled by grey small points in each panel while magenta little squares were used for approximate analytical results. In addition, the blue stars indicate the values of the amplitude *a* of the zero-order approximation.
From the graphs, it is apparent that for a small forcing, $m < 0.05$, the two solutions fit precisely for the entire domain of frequency Ω excepting a small neighbourhood of $\Omega = 0.8$. The agreement continues to be pretty good in the range $m \in [0.05, 0.2]$, but only for those frequencies Ω for which equation (9) has bounded solutions.
Some notable differences, which correspond to a subharmonic resonance, appear near $\Omega \cong 0.27$, for $m > 0.1$. The four panels prove that the component $\varepsilon\Theta$ has a small impact on the global amplitude of oscillation, A_{apx}. The differences between $A_{apx}$and a are practical negligible.
One might think that a simple solution of form

$$\theta(t) = a_{HB}cos(\Omega t + \delta) = A_c\, cos\Omega t + A_s \sin\Omega t$$

obtained, for example, by the harmonic balance method, would represent an alternative to the previous approach. By replacing this assumed solution in the roll equation (9) and matching the coefficients of $cos\Omega t$ and $\sin\Omega t$, the following system in A_c and A_s is obtained:

$$\begin{cases} (k_1 - \Omega^2)A_c + d_1\Omega A_s + \frac{3}{4}(A_c^2 + A_s^2)(k_3 A_c + d_3\Omega^3 A_s) = m \\ -d_1\Omega A_c + (k_1 - \Omega^2)A_s + \frac{3}{4}(A_c^2 + A_s^2)(-d_3\Omega^3 A_c + k_3 A_s) = 0 \end{cases}$$



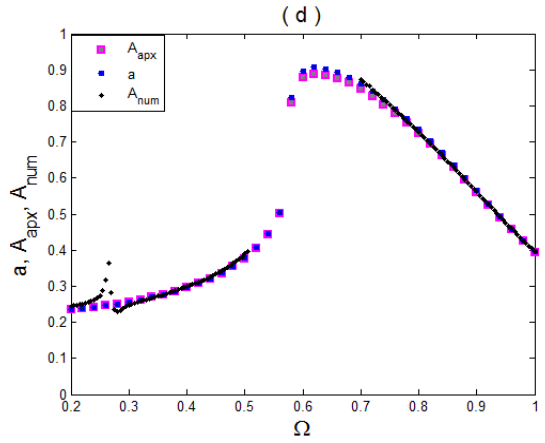


Fig. 4. Comparison of frequency – response curves of equation (9) obtained by numerical integration (A_{num}) and modified multiple scale method (A_{apx}) for:
(a) $m = 0.01$; (b) $m = 0.04$; (c) $m = 0.1$; (d) $m = 0.15$

The values a correspond to the amplitude of zero – order approximation $a\,cos(\Omega t + \psi)$.
Figure 5, which shows a comparison between the amplitude of roll oscillation obtained numerically and that found by solving the previous system, $a_{HB} = \sqrt{A_c^2 + A_s^2}$, shows that the harmonic balance method underestimates the amplitude value, especially in the resonance area and to its right.

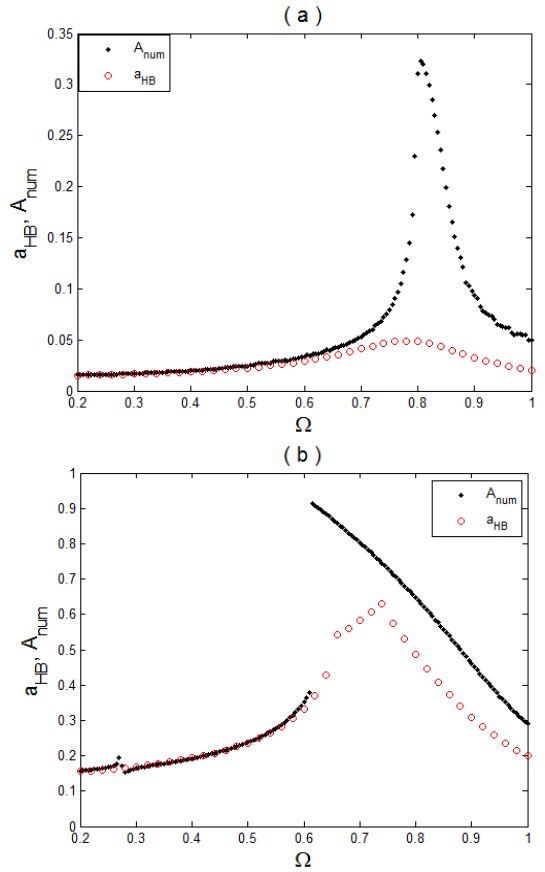
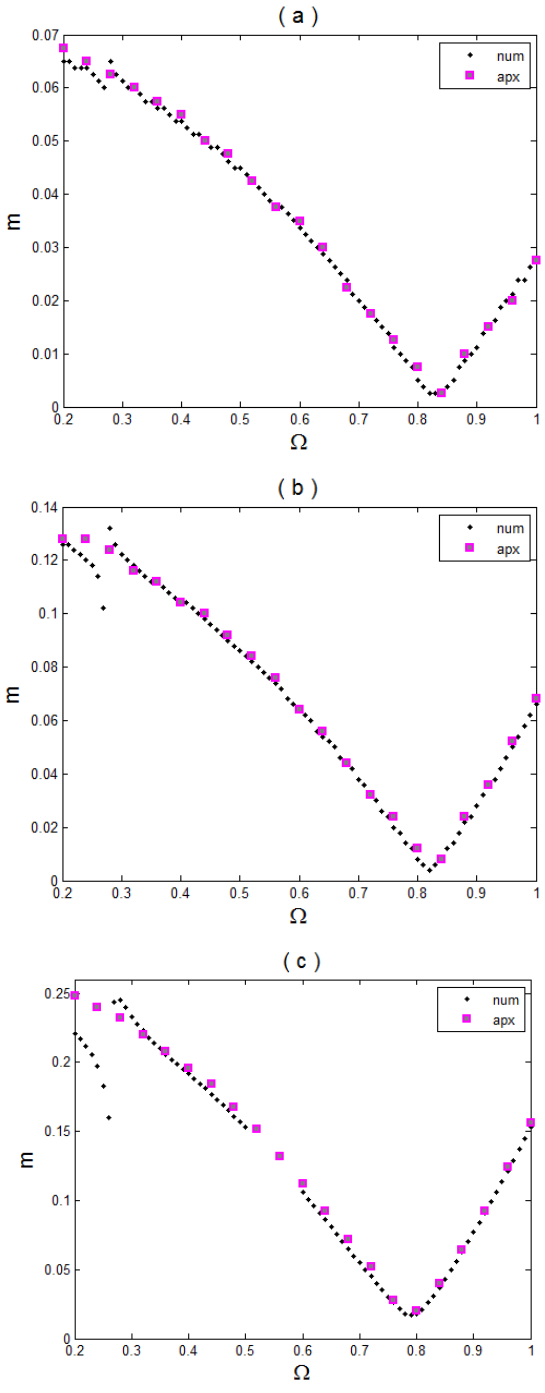


Fig. 5. Comparison of frequency – response curves of equation (9) obtained by numerical integration (A_{num}) and harmonic balance method (a_{HB}) for: (a) $m = 0.01$; (b) $m = 0.1$

In practice, it may be more important to know what is, for a given frequency Ω, the value of the forcing amplitude m for which a certain roll amplitude is reached. Figures 6a to 6c show three such representations (for 0.1 rad, 0.2 rad and 0.4 rad). Again, it is worth highlighting the overlap of the approximate solution over the exact one, except for the secondary resonance area near $\Omega \cong 0.27$. Here, one has to think of another approximate solution, obtained with another technique.
The three panels, linked to figure 1, show that the three curves are approximately parallel to the boundary that separates, in the plane (Ω, m), the domain associated with finite solutions from the one leading to boundless solutions.



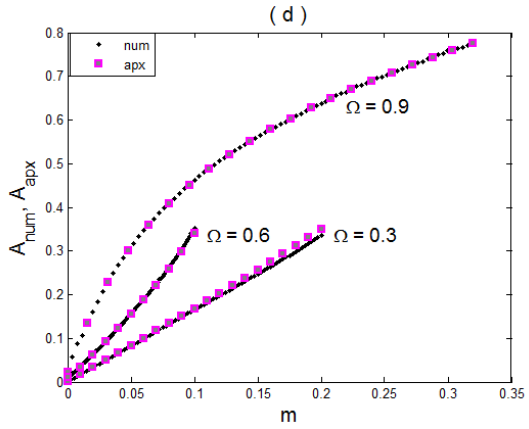


Fig. 6. The constant roll amplitude curves (*a*) $A = 0.1$; (*b*) $A = 0.2$; (*c*) $A = 0.4$., (*d*) Roll amplitude as a function of forcing intensity m for different wave frequencies Ω

It should also be noted that in the neighbourhood of the main resonance, small increases in forcing amplitude m have the effect of significant changes in the roll amplitude, with the danger of reaching the angle of vanishing stability. The same type of information is presented in figure 6*d*.

6. CONCLUSIONS

In the paper, a modified multiple time scale method was employed to solve approximately a nonlinear ship roll equation. Due to the large volume of calculations, the study was limited to determining the solution in the first approximation. The higher order approximations produce only a refinement of the solution, without enriching it qualitatively. The main conclusions of the study are as follows:

a) The algorithm of the method is clear and relatively easy to implement;

b) Even if the nonlinearity, both in terms of damping and restoring, is not small, the first-order solution produced by the approximate method overlaps very well with the numerical one (obtained with *ode45* in Matlab), both in the transition phase and in the stationary one;

c) The method used in the paper provides better results than other existing methods, such as harmonic balance or original multiple time scale technique;

d) The simplicity of the approximate solution (the sum of three harmonic functions) allows both an analysis of the influence of different parameters on the ship response in roll, and a quick graphical representation of important dependencies, such as frequency - response curves or operational limits curves;

e) For high enough forcing amplitudes and near the primary resonance, the perturbation method gives unrealistic periodic solutions (involving roll angles greater than 80 – 90 degrees) while the numerical integration leads to boundless solutions. Close to secondary resonance there is, for large external amplitudes, a noticeable difference between the results obtained by integration and the approximate ones.

In conclusion, the modified multiple time scale method is worth considering in the study of other nonlinear forced oscillatory systems.

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