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FFT-ASSISTED SOLUTION FOR THE EIGENSTRESS PROBLEM IN AN ELASTIC HALF-SPACE

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Abstract: A numerical solution circumventing the mathematical complexity of the fully analytical solution of the eigenstress problem is advanced in this paper, whose strong point is the computational efficiency implied by the convolution and correlation theorems, assisted by the fast Fourier transform. The half-space solution is constructed by superposition of two solutions for the infinitely extended medium and an appropriate boundary value problem. The needed eigenstresses result by superimposing three stress states that can be obtained more easily. The first elastic state places the input eigenstrains in an infinite medium. A second infinite space state has a special distribution of eigenstrains, which geometrically mirrors the input eigenstrains with respect to the half-space boundary. Superposition of the two infinite medium states replicates the problem of the input eigenstrains in the half-space, except for the boundary, where a spurious distribution of normal tractions remains. The latter distribution generates a stress state that needs to be superimposed to the superposition of the first two states, yielding the half-space elastic state. The third state has a well-known solution developed in Contact mechanics, based on the Boussinesq fundamental solutions for the elastic half-space. Scenarios reported in the literature concerning uniform dilatational eigenstrains inside an ellipsoidal inclusion are replicated with the proposed method, validating the newly advanced computer program. New results are then obtained by considering dilatational eigenstrains that decrease proportionally with the ellipsoid radii. The latter scenario is of practical interest because it is met in elastic-plastic contacts under moderate load.
Key words: eigenstrains, eigenstress, fast Fourier transform, convolution, correlation, half-space.

1. INTRODUCTION

Finding the elastic fields (eigenstresses) associated with non-elastic strains such as plastic strains, misfit strains or thermal expansion, generically referred to as eigenstrains, is a fundamental problem in the study of the mechanical properties of solids containing cracks, voids or inclusions. Although the problem has received a great deal of attention [1,2], the number of cases with analytical solutions remains limited. In most situations, the complexity of the mathematical models may discourage integration in engineering applications. A numerical approach, capable of dealing with arbitrary eigenstrain configurations, presents itself as a worthy substitute.

The starting point for the numerical strategy is the analytical expression of stresses due to a cuboidal inclusion in infinite space obtained by Chiu [3]. An extension to the case of subsurface cuboidal regions in a half-space was later achieved [4] with the mirror image method. Two infinite space solutions of two cuboids with specified initial strains were superimposed, and their plane of symmetry was then relieved by the normal stress. The same technique was later employed by Jacq et al. [5] for the elastic-plastic contact problem with residual stresses calculation. For the latter, the plastic volume was replicated by a set of elementary cuboids, each presumed with uniform plastic strains. The method was revisited by Wang and Keer [6], who further implemented the Discrete Convolution Fast Fourier Transform (DCFFT) [7,8] in layers parallel to the half-space border. The inefficiency of the calculations related to the residual stresses was the main limiting factor.

To overcome this limitation, Liu and Wang [9] approached the inclusion problem by superposition, differentiation and integration of Mindlin's [10] Green's functions for the semi-infinite solid. Other authors [11,12] focused on the introduction of 3D algorithms based on the DCFFT to further increase the computational efficiency. In an elastic-plastic contact solver, the iterative approach requires the repeated calculation of the residual stresses, thus the improvement of the algorithmic efficiency of the eigenstress problem is a relevant research direction. This paper reports a new algorithm for the eigenstress problem that incorporates 3D summation techniques based on the DCFFT. Convolutions and correlation are calculated in a 3D hybrid technique based on the DCFFT. The mirror image method is applied to ensure method simplicity. The free surface relief is also achieved with DCFFT-assisted computation.

2. ASSEMBLY OF HALF-SPACE SOLUTION

The eigenstress problem in an elastic half-space is solved by superimposing two solutions for the infinite surrounding matrix and a boundary value problem for the half-space. This procedure was originally advocated by Mura [2], who asserted that a finite body with a traction-free surface can be simulated as an infinitely extended body to which equal but opposite normal and shear stresses are attached on a plane matching the limiting boundary. The spurious stresses compensate for the ones resulting from the infinite space solution. Thus, these stresses create an elastic state that needs to be superimposed to the one corresponding to the infinite surrounding matrix. These summations replicate the elastic state in the half-space eigenstress problem.

This author [13] advanced a numerical solution for the eigenstress problem in an infinite space, capable of finding the stresses induced by an arbitrary distribution of eigenstrains. To this end, superposition was applied to eigenstresses induced by cuboidal regions of uniform eigenstrains. The continuous region with eigenstrains was discretised as a reunion of non-intersecting cuboids, each having the value in the centre as representative for the entire cuboid. In this manner, continuous problem parameters were conceptually replaced by piece-wise constant distributions. In the resulting numerical model, continuous functions of coordinates were substituted by 3D arrays of discrete values. The model equations were then written taking into account the indexes of the centres of the elementary cuboids instead of continuous coordinates.

For a 3D mesh with N_1, N_2, N_3 grids along the three directions of a Cartesian coordinate system, the full space eigenstress tensor $\boldsymbol{\sigma}$ due to the eigenstrains tensor **e**, was expressed [13] as a 3D convolution product:

$$\sigma_{\xi\zeta}(i,j,k) = \sum_{\ell=1}^{N_1}\sum_{m=1}^{N_2}\sum_{n=1}^{N_3} A_{\xi\zeta\varsigma\gamma}(i-\ell, j-m, k-n) \times e_{\varsigma\gamma}(\ell,m,n), \tag{1}$$

in which the fourth order tensor $A_{\xi\zeta\varsigma\gamma}$ expresses the eigenstress observed in the (i,j,k) control point, induced by a cuboid of uniform eigenstress centred in origin.

For the half-space solution, it is convenient to choose the origin of the coordinate system on the half-space boundary, with the x_3-axis pointing inward. A spurious digitised eigenstrain distribution **f** is then assumed in the half-space $x_3 < 0$, with **e** and **f** symmetrically distributed about the x_1x_2 plane, matching the half-space boundary. In every two symmetrical control points, the relation between the two eigenstrains regions is chosen as follows:

$$\begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{12} & f_{22} & f_{23} \\ f_{13} & f_{23} & f_{33} \end{bmatrix} = \begin{bmatrix} e_{11} & e_{12} & -e_{13} \\ e_{12} & e_{22} & -e_{23} \\ -e_{13} & -e_{23} & e_{33} \end{bmatrix}. \tag{2}$$

If the two eigenstrain distributions **e** and **f** are assumed in full space, the eigenstress induced by each of them can be calculated with (1). On the plane matching the half-space boundary, the shear stresses induced by each cuboid from the **e** are cancelled by the shear stresses induced by its symmetrical counterpart from **f**. Consequently, after superposition, the boundary is devoid of shear stresses and has only normal tractions. Thus, we now have a full space solution, consisting in the superposition of elastic states due to the eigenstrains distributions **e** and **f**, and a boundary that needs to be relieved of normal tractions. By denoting the latter with $\sigma_{33}^{(0)}$, we can now express the eigenstresses in the half-space limited by the x_1x_2 plane of symmetry as the superposition of three elastic fields:

$$\sigma_{\xi\zeta}(i,j,k) = \sigma_{\xi\zeta}^{(\mathbf{e})}(i,j,k) + \sigma_{\xi\zeta}^{(\mathbf{f})}(i,j,k) - \sigma_{\xi\zeta}^{(x_3>0)}(\sigma_{33}^{(\mathrm{b})}). \tag{3}$$

The assembly of these elastic states whose superposition replicates the considered problem is depicted in figure 1, with each representation related to a term from (3). For brevity, the reunion of cuboids resulting from the digitization of the continuous distribution of eigenstrains was represented by a single cuboid. The plane $x_3 = 0$ is the plane of symmetry for the eigenstrains distributions **e** and **f**. The first addend is the full-space solution for the original eigenstrain distribution, and can be computed with (1). The second one is also a full-space solution, but differs in form from (1) due to the negative sign of the eigenstrain coordinate in the direction normal to the plane of symmetry. Thus, by adapting (1), the second addend can be expressed as:

$$\sigma_{\xi\zeta}^{(\mathbf{f})}(i,j,k) = \sum_{\ell=1}^{N_1}\sum_{m=1}^{N_2}\sum_{n=1}^{N_3} A_{\xi\zeta\varsigma\gamma}(i-\ell, j-m, k+n) \times f_{\varsigma\gamma}(\ell,m,n). \tag{4}$$

It is noteworthy that, as both addends are full space solutions, the influence function $A_{\xi\zeta\varsigma\gamma}$ remains the same, but its arguments are different for the direction of $\vec{x}_3$.

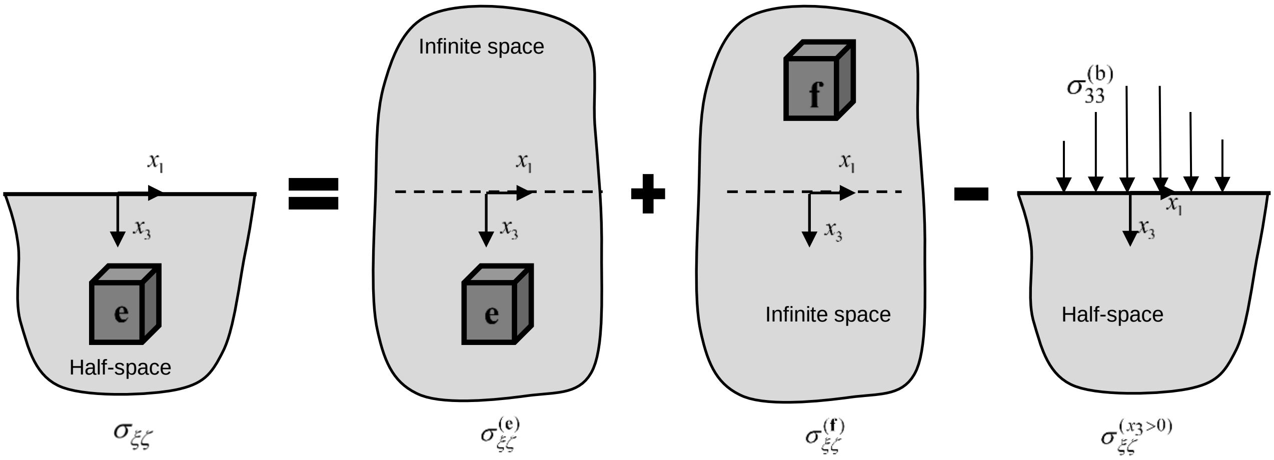

Fig. 1. Superposition of states replicating the half-space problem conditions

In [13], the 3D convolution operation (1) was performed via the DCFFT algorithm [7,8]. The latter is built on the convolution theorem, stating that a convolution in the time/space domain can be calculated, in the Fourier domain, as an element-by-element product between the transformed convolution factors. The latter property decreases considerably the computational complexity of (4), from $O(N_1^2 N_2^2 N_3^2)$ using standard multi-summation, to $O(N_1 N_2 N_3 \log N_1 N_2 N_3)$.

Relation (4) is a convolution product with respect to directions of $\vec{x}_1$ and $\vec{x}_2$, and a correlation product in the direction of $\vec{x}_3$. The 3D DCFFT can be adapted to allow for hybrid convolution-correlation products. The base for this extension is the correlation theorem, stating that a correlation can be calculated similar to a convolution, once the order of elements in the input signal is inverted. The flow-chart for the mixed convolution-correlation algorithm tasked with the simultaneous evaluation of (1) and/or (4) is depicted in figure 2. A detailed description of the DCFFT is beyond the point of this paper, but can be found elsewhere [7,8]. The two full space solutions (1) and (4) yield a normal stress on the half-space border x_1x_2:

$$\begin{aligned}\sigma_{33}^{(\mathrm{b})}(i,j,1) &= \sum_{\ell=1}^{N_1}\sum_{m=1}^{N_2}\sum_{n=1}^{N_3} A_{\xi\zeta\varsigma\gamma}(i-\ell, j-m, -n) \times e_{\varsigma\gamma}(\ell,m,n)\ldots \\ &+\sum_{\ell=1}^{N_1}\sum_{m=1}^{N_2}\sum_{n=1}^{N_3} A_{\xi\zeta\varsigma\gamma}(i-\ell, j-m, n) \times f_{\varsigma\gamma}(\ell,m,n).\end{aligned} \tag{5}$$

By treating the x_1x_2 plane as the border of an elastic half-space resting in the domain $x_3 > 0$, the normal stresses from (5) induce in the aforementioned half-space a stress field $\sigma_{\xi\zeta}^{(x_3>0)}(\sigma_{33}^{(\mathbf{b})})$. To achieve the problem solution, the latter field should be subtracted from the sum of (1) and (4). It should be noted that Einstein summation convention is employed in (5) to account for the contribution of all plastic strain tensor components, the role of the dummy variable (or the source point) being assumed by the (ℓ,m,n) set of indexes. It is transparent from (5) that the source point passes through the whole computational domain and not only the regions with eigenstrains. This feature, apparently increasing the computational effort, is required for implementation of the DCFFT. The detailed calculation of the third addend $\sigma_{\xi\zeta}^{(x_3>0)}$ is discussed in the next section.

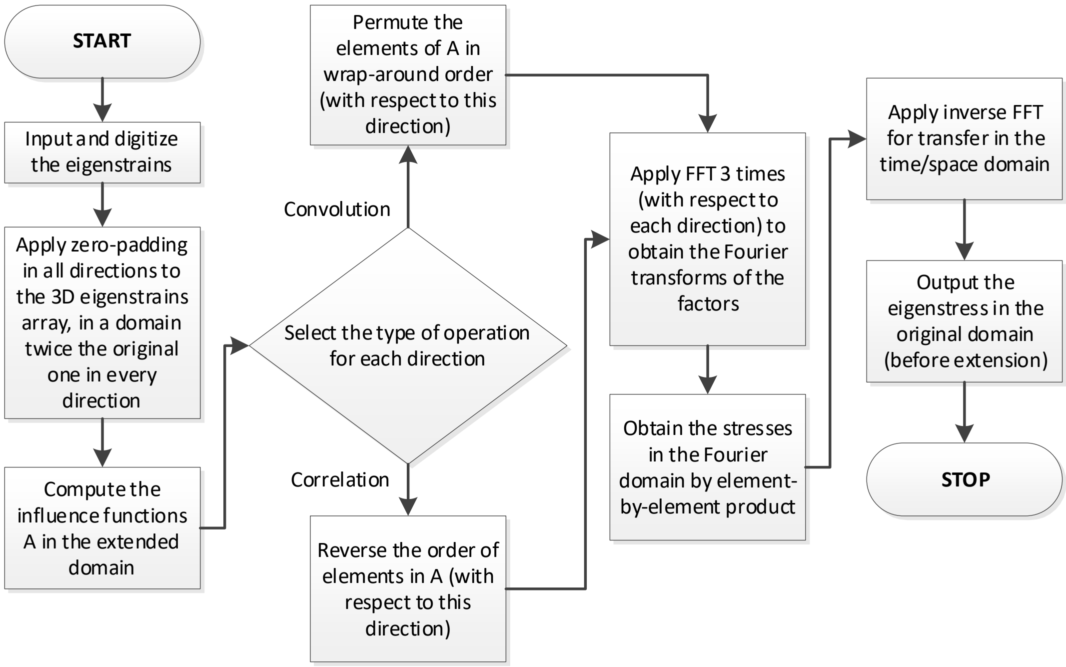

Fig. 2. Flow-chart of the algorithm for the assessment of full space stresses (1) and (4)

3. NUMERICAL SOLUTION FOR THE BOUNDARY CONDITION

The geometrical configuration depicted in figure 1, together with the choice of eigenstrains in **f** according to (2), lead to vanishing shear tractions on the plane of symmetry $x_3 = 0$, matching the half-space boundary. However, the fictitious normal traction denoted by $\sigma_{33}^{r(b)}$ in figure 1 and in equation (3) have to be subtracted from the summation of the infinite space solutions to replicate the free boundary condition.

The calculation of the stress state induced in the elastic half-space by imposed digitized normal tractions is established by the theory of Contact mechanics. To this end, influence functions that are indefinite integrals of the Green's functions for the elastic half-space are employed. Boussinesq, cited in [14], obtained analytically the stress field in an elastic half-space related to a unit concentrated force perpendicular to the half-space border. Integration of the latter solutions over a rectangular domain yields the stresses due to a uniform rectangular distribution of normal tractions.

By denoting with Δ_1 and Δ_2 the grid steps along directions of $\vec{x}_1$ and of $\vec{x}_2$, respectively, the double integrals $b_{\zeta\xi}$ of the Boussinesq stress solutions can be expressed as:

$$b_{11}(x_1,x_2,x_3) = \frac{x_1x_2x_3}{r(x_1^2+x_3^2)} + (1-2\nu)\left(\tan^{-1}\left(\frac{x_1}{x_2}\right) - \tan^{-1}\left(\frac{x_1x_3}{x_2r}\right)\right) - \tan^{-1}\left(\frac{x_1x_2}{x_3r}\right); \quad (6)$$

$$b_{22}(x_1,x_2,x_3) = b_{11}(x_2,x_1,x_3); \quad (7)$$

$$b_{33}(x_1,x_2,x_3) = -\tan^{-1}\left(\frac{x_1x_2}{x_3r}\right) - \frac{x_1x_2x_3(r^2+x_3^2)}{r(x_1^2+x_3^2)(x_2^2+x_3^2)}; \quad (8)$$

$$b_{12}(x_1,x_2,x_3) = (2\nu-1)\log(r+x_3) - x_3/r\,; \quad (9)$$

$$b_{23}(x_1,x_2,x_3) = x_1x_3^2 \big/ [r(x_2^2+x_3^2)]\,; \quad (10)$$

$$q_{31}(x_1,x_2,x_3) = x_2x_3^2 \big/ [r(x_1^2+x_3^2)]\,, \quad (11)$$

with $r = \sqrt{x_1^2+x_2^2+x_3^2}$. The needed influence functions then result as:

$$B_{\zeta\xi}(i,j,m) =$$

$$\frac{1}{2\pi}\begin{pmatrix} b_{\zeta\xi}(x_1(i)+\frac{\Delta_1}{2}, x_2(j)+\frac{\Delta_2}{2}, x_3(m)) + b_{\zeta\xi}(x_1(i)-\frac{\Delta_1}{2}, x_2(j)-\frac{\Delta_2}{2}, x_3(m))... \\ -b_{\zeta\xi}(x_1(i)+\frac{\Delta_1}{2}, x_2(j)-\frac{\Delta_2}{2}, x_3(m)) - b_{\zeta\xi}(x_1(i)-\frac{\Delta_1}{2}, x_2(j)+\frac{\Delta_2}{2}, x_3(m)) \end{pmatrix}. \quad (12)$$

Using the same discretization as for the infinite space solution, the in-depth stresses due to a piece-wise constant distribution of surface normal tractions can be obtained by superposition:

$$\sigma_{\xi\zeta}^{(x_3>0)}(i,j,m) = \sum_{k=1}^{N_1}\sum_{\ell=1}^{N_2} B_{\xi\zeta}(i-k, j-\ell, m)\cdot\sigma_{33}^{(b)}(k,\ell,1), \quad (13)$$

where $i=1..N_1$, $j=1..N_2$, $m=1..N_3$, $\xi,\zeta=1,2,3$. The index "1" is used to suggest that, although relation (5) generates a 3D stress array, only the first slice (a 2 2D array) is needed in the second term of the convolution (13), whereas in the first term, the index $m=1..N_3$ traverses all depth grids. Thus, (13) is only a 2D convolution product about the directions of $\vec{x}_1$ and $\vec{x}_2$, as the input distribution is a matrix describing the half-space boundary stresses, assembled from the upper faces of all cuboids $(k,\ell,1)$, with $i=1..N_1$ and $j=1..N_2$.
The method applied in this section implies both a discretization and a truncation error. The discretization error is linked to the model discretization substituting continuous parameters with piece-wise constant distributions, and can be reduced by using finer meshes. On the other hand, the truncation error stems from the calculation of normal traction $\sigma_{33}^{(b)}$ only inside the considered computational domain. Thus, the method precision can be affected if the fictitious normal traction does not decay rapidly enough towards the grid ends. The choice of a larger computational domain, exceeding with a safe margin the volume with eigenstrains, should preserve the method precision.
Regarding the computational efficiency of the proposed method, the treatment of (13) by DCFFT assures that the optimal multi-summation technique is used. 2D DCFFT is applied simultaneously in N_3 layers of constant depth, and the resulting computational complexity is $O(N_1N_2N_3\log N_1N_2)$.

4. METHOD VALIDATION AND CALCULATION EXAMPLES

Uniform dilatational eigenstrains (i.e., $e_{ij}=\delta_{ij}\theta$, with δ_{ij} the Kronecker delta and θ a fixed strain) are first considered inside an ellipsoidal inclusion of half-axes a_i along direction of $\vec{x}_i, i=1,2,3$. The depth of the ellipsoid center measured from the half-space boundary is denoted by h, and $a_1/a_2=1$, while the ratio a_1/a_3 is varied. Dimensionless coordinates are defined as ratio to a_3, and dimensionless eigenstresses as ratio to the stress $\sigma_0=2\mu\theta(1+\nu)/(1-\nu)$, with μ and ν the elastic parameters of the considered material (it is reminded that the inclusion retains the elastic parameters of the surrounding matrix).
A cuboidal computational domain is meshed into $128\times128\times128$ grids. The predictions of the computer program based on the newly proposed method, depicted in figures 3-6, agree with the results presented, [2].
Additional simulations consider a spherical inclusion with dilatational eigenstrains that decrease proportionally with the radius of the sphere. The maximum value in the centre is

$$e_{ij}=\delta_{ij}\theta, \quad (14)$$

and decreases to zero on the boundary of the sphere. The predicted eigenstresses are depicted in figures 7 and 8.
The presented simulations prove that the newly proposed method is well adapted for the considered type of problem and can be used to derive results beyond those existing in the literature.

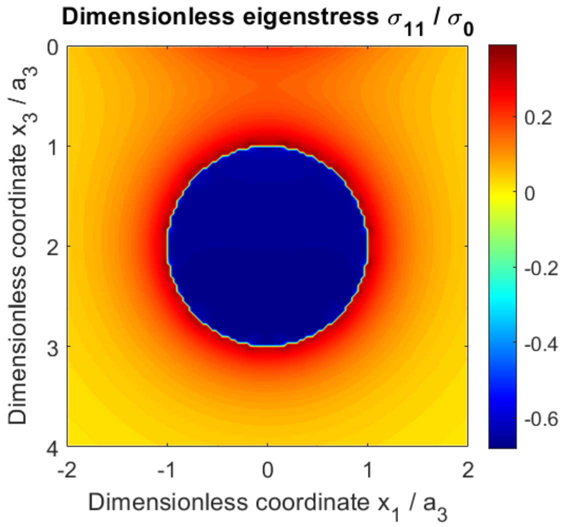

Fig. 3. Iso-contours of dimensionless eigenstress σ_{11}/σ_0, in the case $a_1/a_3 = 1$, $h/a_3 = 2$

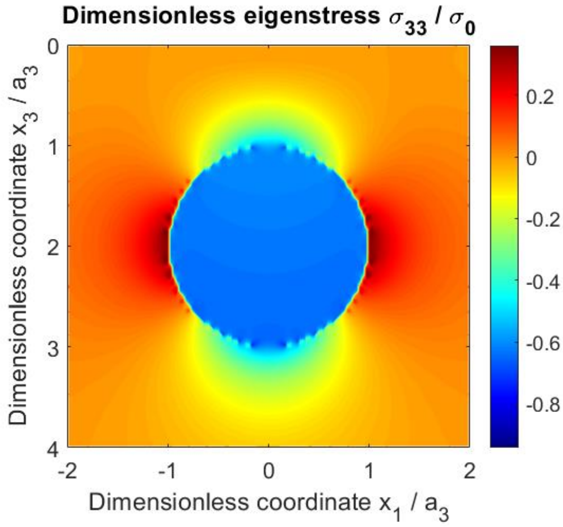

Fig. 4. Iso-contours of dimensionless eigenstress σ_{33}/σ_0, in the case $a_1/a_3 = 1$, $h/a_3 = 2$

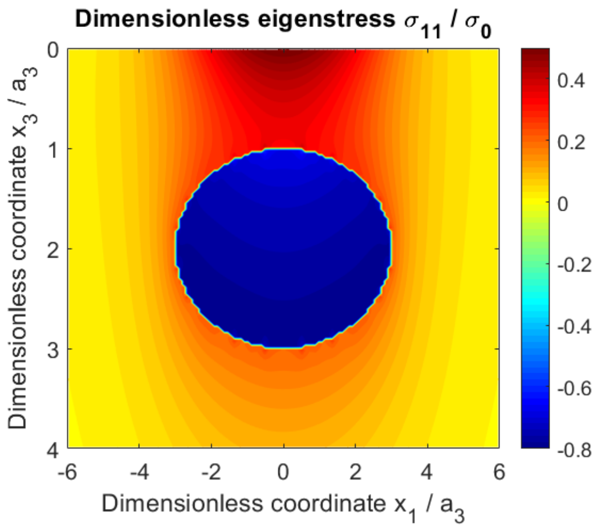

Fig. 5. Iso-contours of dimensionless eigenstress σ_{11}/σ_0, in the case $a_1/a_3 = 3$, $h/a_3 = 2$

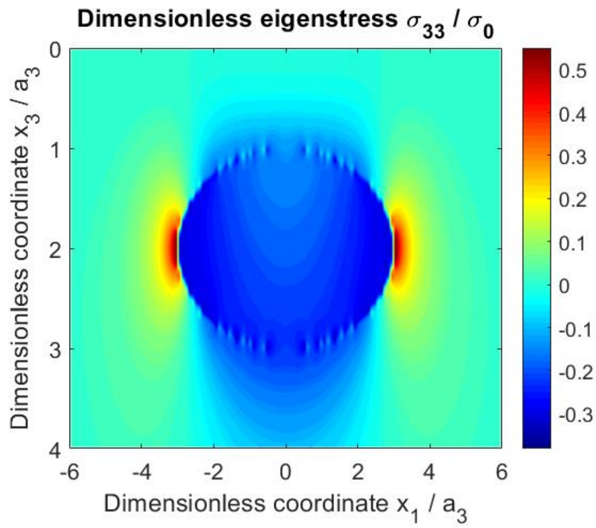

Fig. 6. Iso-contours of dimensionless eigenstress σ_{33}/σ_0, in the case $a_1/a_3 = 3$, $h/a_3 = 2$

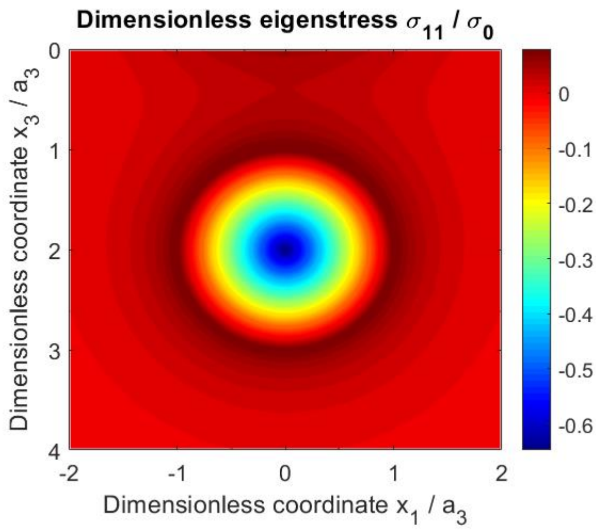

Fig. 7. Iso-contours of dimensionless eigenstress σ_{11}/σ_0, in the case $a_1/a_3 = 1$, $h/a_3 = 2$, non-uniform strains

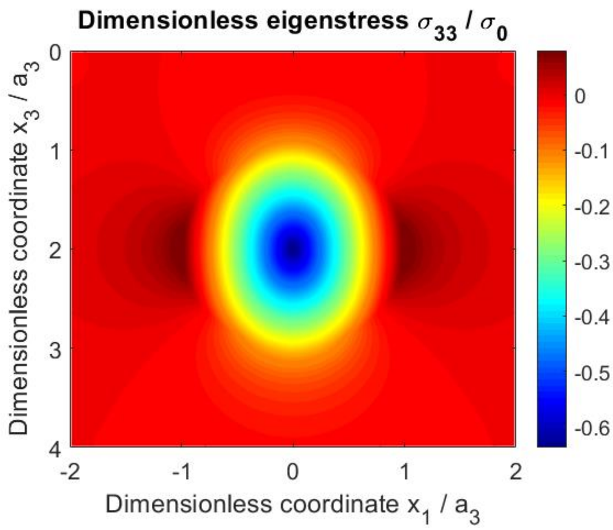

Fig. 8. Iso-contours of dimensionless eigenstress σ_{33}/σ_0, in the case $a_1/a_3 = 1$, $h/a_3 = 2$ non-uniform strains

5. CONCLUSIONS

An efficient and robust numerical method for the eigenstress problem in a half-space is proposed in this paper, marking an additional reduction in the computational effort, which remains to date the most important restrictive factor for the problem solution.
The mirror image technique is applied in the inclusion problem solving, by assembling two infinite space solutions and one half-space elastic filed that reproduces the traction-free boundary conditions.
The calculation of the infinite space solution is achieved with the convolution and the correlation theorems assisted by the fast Fourier transform applied in 3D, resulting in an important decrease of computational complexity. The traction-free boundary sub-problem is solved using numerical techniques originally developed for Contact mechanics, accelerated by 2D FFT.
The presented calculation examples are a good match to existing solutions for regularly shaped inclusions.
The method can be applied to predict the residual stresses in materials after plastic deformation based on a known distribution of plastic strains. In particular, the method can assist the solution of the mechanical contact problem in the elastic-plastic domain with finer meshes and thus greater accuracy.

REFERENCES

1. Mura T., (1988). *Inclusion Problem*, ASME Applied Mechanics Review, 41, 15-20.
2. Mura T., (1987). Micromechanics of Defects in Solids, The Hague: Martinus Nijhoff Publishers.
3.Chiu Y P., (1977). *On the Stress Field Due to Initial Strains in a Cuboid Surrounded by an Infinite Elastic Space*, ASME Journal of Applied Mechanics, 44, 587-590.
4. Chiu Y P., (1978)., *On the Stress Field and Surface Deformation in a Half Space with Cuboidal Zone in Which Initial Strains Are Uniform*, ASME Journal of Applied Mechanics, 45(2), 302-306.
5. Jacq C, Nelias D, Lormand G, Girodin D., (2002), *Development of a Three-Dimensional Semi-Analytical Elastic-Plastic Contact Code* ASME Journal of Tribology, 124, 653-667.
6. Wang F, Keer L M (2005) *Numerical Simulation for Three Dimensional Elastic-Plastic Contact with Hardening Behavior*, ASME Journal of Tribology, 127, 494-502.
7. Liu S B, Wang Q, and Liu G (2000), *A versatile method of discrete convolution and FFT (DC-FFT) for contact analyses*, Wear, 243(1–2), 101–110.
8. Liu S, Wang Q., (2001), *Studying Contact Stress Fields Caused by Surface Tractions with a Discrete Convolution and Fast Fourier Transform Algorithm*, ASME J. Tribol., 124, 36-45.
9. Liu S, Wang Q., (2005). *Elastic Fields due to Eigenstrains in a Half-Space*, ASME Journal of Applied Mechanics, 72, 871-878.
10. Mindlin R D., (1936). *Force at a Point in the Interior of a Semi-Infinite Solid*, Physics, 7, 195-202.
11. Zhou K, Chen W W, Keer L M, Wang Q J., (2009). *A Fast Method for Solving Three-Dimensional Arbitrarily Shaped Inclusions in a Half-Space*, Computer Methods in Applied Mechanics and Engineering, 198, 885-892.
12. Spinu S., (2014). *A Robust Algorithm to Compute Residual Stresses in Elastic-Plastic Contacts*, Journal of the Balkan Tribological Association, 20, 63-74.
13. Spinu S., (2023). *FFT-Assisted Solution for the Eigenstress Problem in an Infinite Elastic Medium*, International Journal of Modern Manufacturing Technologies, 15, 141-147.
14. Johnson K L., (1996). *Contact Mechanics*, Cambridge: University Press.

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