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RESIDUAL DISPLACEMENT DUE TO ARBITRARY PLASTIC STRAINS

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Abstract: The displacement of the boundary of an elastic body in which inelastic strains exist or initiate due to various conditions can affect the intended functioning of the machine element and thus the service life of the assembly. Plastic strains are often initiated when the body is loaded beyond the elasticity limit of its material. The subsequent deformation of the boundary may affect the interaction with the coupling elements, resulting in further perturbations of the contact conditions. This is especially true when load is transmitted through a small contact region, such as in gears or bearings. Assessment of this permanent deformation is thus essential to predict and prolong the service life of various machine elements working under severe contact loads. The mathematical model employed in this work is built around the Betti's reciprocal theorem, which exposes a property of the Elasticity equations in the presence of inelastic strains. The resulting equations prove that the residual displacement due to plastic strains in the bulk of the body can be expressed by superposition of effects, as a convolution product. The Green's functions for the latter convolution are integrated to derive the influence coefficients needed in the numerical approach. The model discretisation is rectangular, with the volume of plastic strains approximated with a reunion of adjacent cuboids with uniform strains. The contributions of each elementary cuboid to the displacement components in a specific point from the boundary, i.e., the influence coefficients, are derived and expressed analytically. With these elements, an acceleration technique for the computation of elastic displacement can be extended to residual displacement. Calculations are performed with the aid of a Matlab computer program developed based on the newly derived relations. Five configurations of plastic strains are considered and the program predictions are compared with existing results from the literature. The good agreement gives confidence in the advanced computer tool and encourages its implementation in a more complex program for the simulation of the elastic-plastic contact.
Key words: plastic strains, residual displacement, reciprocal theorem, fast Fourier transform, convolution.

1. INTRODUCTION

Knowledge of residual displacement due to inelastic strains in the body of a machine element, regardless of their origin, is an important topic in mechanical engineering, as the latter displacement modify geometry as thus can affect the intended functioning of the product. Phase transformation, thermal expansion, plastic strains, and misfit strains are well-known examples of such inelastic strain, referred to in the literature [1] as eigenstrains. Plastic strains arise in many machine elements under contact load, especially when the contact is non-conforming (i.e., the initial contact area vanish, e.g. point of line contacts) and load is transmitted through a limited contact area established from the deformation of the softer contacting element. Due to regular inherent vibrations or steep load gradients, the elastic limit of the softer material may be surpassed and plastic yield initiated, leading to changes in the product functioning. Fretting is now well recognized as one of the major causes of damage of machine elements functioning under contact load. Prediction of the contact behavior in the presence of plastic strains, which requires the knowledge of the residual displacement, is of paramount importance to prolong the service life of mechanical contacts.

Although the commercial finite element software can predict an elastic-plastic contact process [2,3], the approach is considered as unsuitable to contact problems, because the meshing of the bulk body is required, whereas contact stresses arise only in a vicinity of the initial contact area (of vanishing dimensions in case of non-conforming contacts). On the other hand, in that region the stress gradients are very steep, and a very fine mesh is needed to capture the specifics of the contact state. Moreover, plasticity as a dissipative process requires the simulation of the entire loading history. For this reasons, finite element analysis of contact processes is considered prohibitive even for modern computers, and important research efforts were conducted to solve the contact problem based on equations from elasticity theory developed for the elastic half-space, coupled with the Boundary Element Method [4]. This resulted in a method known as semi-analytical (SAM), pioneered by Mayeur [5] for the two-dimensional elastic-plastic rough contact. Betti's reciprocal theorem and BEM were combined in an algorithm capable of predicting the contact pressure based on the mutual interaction between residual displacement and contact pressure. Prior to this, the approach of limiting the contact pressure to three times the material hardness was used. Whereas this simplification may be accurate for materials exhibiting elastic-perfectly-plastic behavior, it cannot predict the hardening of the material.

The extension of Mayeur's work to three dimensions was achieved by Jacq et al. [6], who derived the problem dependencies in the three-dimensional space and applied methods of computation acceleration based on the fast Fourier transform. Many researchers [7-15] improved on this original design, either by improving the computational efficiency, or by considering additional plastic laws behaviors or more complex contact scenarios. The work reported in this paper is based on SAM and shares the same framework. The derivation of displacement is reduced to a convolution product for which an acceleration technique is applied. This modern approach to convolution calculation assures that the best available computational efficiency is achieved. A computationally efficient Matlab computer tool is created and its predictions are matched against existing results.

2. THEORETICAL BACKGROUND

Superposition principle cannot be generally applied to plasticity problems because the linearity between cause and effect is broken in the frame of plasticity, in which the final state depends on the entire history of the process (i.e., a hysteretic behaviour). However, Betti's reciprocal theorem provides grounds for applying superposition under certain limiting conditions. The treatment of plasticity problems via Betti's theorem pioneered with the works of Brebbia [4] and Mayeur [5], whereas Jacq et al. [6] established the first full-working model for the three-dimensional elastic-plastic contact. An outline of the Betti's application to residual displacement is presented in this section, establishing the base equations comprising the continuous model of the problem.

Let us consider two independent elastic states $S_p:(\mathbf{u},\boldsymbol{\varepsilon},\boldsymbol{\sigma}, f_i)$ and $S_e:(\mathbf{u}^*,\boldsymbol{\varepsilon}^*,\boldsymbol{\sigma}^*, f_i^*)$ in a three-dimensional domain Ω bounded by a surface Γ, where $\mathbf{u}$ denotes the displacement, $\boldsymbol{\varepsilon}$ the total strain, $\boldsymbol{\sigma}$ the elastic stress and f_i the applied body force. The displacement fields $\mathbf{u}$ and $\mathbf{u}^*$ are continuous with the first and second derivatives as continuous functions. Whereas S_e is a purely elastic state, S_p exists with inelastic initial strains $\boldsymbol{\varepsilon}^0$ that can be of various origin (phase transformation, thermal expansion, plastic strains or misfit strains, i.e. eigenstrains [1]). The stress and strain fields in the two states can be expressed as:

$$S_p:\begin{cases}\varepsilon_{ij}=\frac{1}{2}\left(u_{i,j}+u_{j,i}\right);\\ \sigma_{ij}=M_{ijk\ell}\left(\varepsilon_{k\ell}-\varepsilon_{k\ell}^0\right);\end{cases}\qquad S_e:\begin{cases}\varepsilon_{ij}^*=\frac{1}{2}\left(u_{i,j}^*+u_{j,i}^*\right);\\ \sigma_{ij}^*=M_{ijk\ell}\varepsilon_{k\ell}^*,\end{cases}\tag{1}$$

where $M_{ijk\ell}=\lambda\delta_{ij}\delta_{k\ell}+\mu(\delta_{ik}\delta_{j\ell}+\delta_{i\ell}\delta_{jk})$ is the elastic stress tensor, with λ and μ the Lamé's constants and δ the Kronecker delta. Indexes i,j,k,ℓ range from 1 to 3 and Einstein summation convention over repeated indices is applied.

By evaluating the product $\sigma_{ij}\varepsilon_{ij}^*$, one can verify that $\sigma_{ij}\varepsilon_{ij}^*=\left(\varepsilon_{ij}-\varepsilon_{ij}^0\right)\sigma_{ij}^*$, which can be integrated over Ω, where $\mathbf{u}$ and $\mathbf{u}^*$ are continuous together with the first two derivatives, yielding:

$$\int_\Omega \sigma_{ij}\varepsilon_{ij}^* d\Omega=\int_\Omega \varepsilon_{ij}\sigma_{ij}^* d\Omega-\int_\Omega \varepsilon_{ij}^0\sigma_{ij}^* d\Omega.\tag{2}$$

Using the Gauss-Ostrogradsky theorem, the left-hand side of (2) can be expressed as:

$$\int_\Omega \sigma_{ij}\varepsilon_{ij}^* d\Omega=-\int_\Gamma u_i^*\sigma_{ij}n_j d\Gamma+\int_\Omega f_i u_i^* d\Omega,\tag{3}$$

where n_j is normal to Γ and pointing inwards (toward Ω). Further on, the first term in the right-hand side of (2) can be written as:

$$\int_{\Omega} \varepsilon_{ij}\sigma_{ij}^{*} d\Omega = -\int_{\Gamma} u_i\, \sigma_{ij}^{*} n_j d\Gamma + \int_{\Omega} f_i^{*} u_i\, d\Omega\,. \tag{4}$$

By plugging (3) and (4) into (2) one can obtain the general form of the reciprocal theorem, which in fact expresses an inherent property of the elasticity equations in the presence of initial inelastic strains:

$$-\int_{\Gamma} u_i^{*}\sigma_{ij} n_j d\Gamma + \int_{\Omega} f_i u_i^{*} d\Omega = -\int_{\Gamma} u_i\, \sigma_{ij}^{*} n_j d\Gamma + \int_{\Omega} f_i^{*} u_i\, d\Omega - \int_{\Omega} \varepsilon_{ij}^{0}\sigma_{ij}^{*} d\Omega\,. \tag{5}$$

In particular, one can consider that the initial strains are of plastic nature, thus $\boldsymbol{\varepsilon}^0 = \boldsymbol{\varepsilon}^p$ and $tr(\boldsymbol{\varepsilon}^p) = 0$, as the plastic strains are of deviatoric nature (i.e., incompressible). Moreover, if $f_i = 0$ in the state S_p:

$$0 = -\int_{\Gamma} u_i\, \sigma_{ij}^{*} n_j d\Gamma + \int_{\Omega} f_i^{*} u_i\, d\Omega - 2\mu \int_{\Omega} \varepsilon_{ij}^{p}\varepsilon_{ij}^{*} d\Omega\,. \tag{6}$$

If further on S_e is chosen so that $f_i^{*} = 0$, but with a surface traction distribution of unity magnitude over a subdomain $\Gamma_C \subseteq \Gamma$:

$$\begin{cases} \sigma_{ij}^{*} n_j = -p_i^{*}, & \text{on } \Gamma_C; \\ \sigma_{ij}^{*} n_j = 0, & \text{on } \Gamma - \Gamma_C, \end{cases} \tag{7}$$

one can further obtain that:

$$-\int_{\Gamma} u_i\, \sigma_{ij}^{*} n_j d\Gamma = \int_{\Gamma} u_i\, p_i^{*} d\Gamma = u_k^{r}\,, \tag{8}$$

or, by employing a notation based on a calculation point *A* and an integration point *M*, with $A, M \in \Gamma$:

$$u_\ell^{r}(A) = 2\mu \int_{\Omega_p} \varepsilon_{ij}^{p}(M)\varepsilon_{kij}^{*}(M, p_k^{*}(A)) d\Omega\,. \tag{9}$$

In the latter, $\Omega_p \subseteq \Omega$ is the subdomain with plastic strains, *A* is a surface point in which a unity traction along $\vec{x}_k$ is applied, denoted by $p_k^{*}(A)$, and thus $\varepsilon_{kij}^{*}(M, p_k^{*}(A))$ is the *ij* component of the strain tensor induced by $p_k^{*}(A)$ in an integration volume point *M*. By varying *A* on Γ and *k* over the three directions, (9) yields the surface displacement due to a distribution of volume plastic strains located in $\Omega_p \subseteq \Omega$. It is noteworthy that $\varepsilon_{kij}^{*}(M, p_k^{*}(A))$ refers to a purely elastic state and thus can be computed in the frame of elasticity. By using coordinates instead of point marks, (9) can be re-written as:

$$u_k^{r}(x_1, x_2) = 2\mu \int_{\Omega_p} \varepsilon_{ij}^{p}(x_1', x_2', x_3')\varepsilon_{kij}^{*}(x_1' - x_1, x_2' - x_2, x_3') d\Omega\,, \tag{10}$$

where $\varepsilon_{kij}^{*}(x_1' - x_1, x_2' - x_2, x_3')$ is the elastic strain induced in the point (x_1', x_2', x_3') by a unit concentrated force acting in the point $(x_1, x_2, 0)$ along $\vec{x}_k$.

Relations (9) or (10) authorise the use of superposition principle in the derivation of residual displacement. Therefore, a numerical approach becomes convenient, in which contributions from elementary plastic strains are superimposed leading to the global residual displacement. A discrete counterpart of the continuous equations (9) or (10) is proposed and a computational technique is advanced in the following section.

3. MODEL DISCRETIZATION AND COMPUTATIONAL TECHNIQUE

Considering the potential use of the discussed computational method, the algorithm should be designed with respect to the specifics of the elastic-plastic contact solvers [7-15], which are all iterative. The need for the reproduction of the path (i.e., the loading history), and the complexity of the mathematical models require models whose solution is achieved by trial-and-error, i.e. an initial approximation is improved though multiple iterations until the model equations are verified to a smaller error. In case of elastic-plastic contacts, the iterated parameter is the plastic strain distribution, and at each iteration a new residual displacement field is computed. Thus, a method is needed to calculate the displacement due to known, but otherwise arbitrary, distributions of plastic strains. Such a task can only be accomplished by a numerical approach, involving a model discretization.

The continuous volume distribution of plastic strains is simulated by a reunion of adjacent cuboids in which all plastic strain tensor components are assumed uniform. In the discrete problem model, each cuboid contributes with a single value (for each tensor component), value that is calculated according to existing analytical solutions by replacing the cuboid with a single point laying in the cuboid centre. In this manner, continuous distributions are approximated by sets of discrete values whose number equal the number of elementary cuboids. As in all numerical treatments, the finer the mesh, the better the approximation, but the greater the computational cost.

For the problem of residual displacements, it can be seen from equation (10) that the observation domain is two-dimensional (described by coordinates (x_1, x_2)), whereas the integration point of coordinates (x_1', x_2', x_3') covers a three-dimensional domain. The computational effort for this type of superposition is non-trivial and acceleration methods are needed to assure that the best precision is attained. By interchanging the coordinates of the computation and of the observation points in the expression of ε_{kij}^* from relation (10), the latter becomes a two-dimensional convolution with respect to directions x_1 and x_2. The method of Discrete Convolution Fast Fourier Transform (DCFFT) [16,17] was lately used for the rapid and precise calculation of discrete convolutions. To apply the latter method, the model discretisation is set to cover a cuboidal region, and the elementary cuboids have the same side-lengths in the three directions, $(\Delta_1, \Delta_2, \Delta_3)$, but the inequality $\Delta_1 \neq \Delta_2 \neq \Delta_3$ is allowed. In other words, the discretisation is uniform, but the step may vary from one direction to the other. The coordinate-based notation can be replaced in (10) with indexes locating the position of each elementary cuboid inside the matrix. A discrete counterpart of (10) can be written as:

$$u_k^r(i,j,0) = 2\mu \sum_{(\ell,m,n)\in\Omega_p} \left(\varepsilon_{\zeta\xi}^p(\ell,m,n) \int\limits_{x_3(n)-\Delta_3/2}^{x_3(n)+\Delta_3/2} \int\limits_{x_2(m)-\Delta_2/2}^{x_2(m)+\Delta_2/2} \int\limits_{x_1(\ell)-\Delta_1/2}^{x_1(\ell)+\Delta_1/2} G_{\zeta\xi}\left(\ell - x_1(i), m - x_2(j), n\right) dx_1'\, dx_2' dx_3' \right), \quad (11)$$

in which i, j, ℓ, m, n are integers ranging from 1 to the number of cuboids on each direction, and the zero in $u_k^r(i,j,0)$ suggests that the displacement of the limiting surface (i.e., $x_3 = 0$) is computed. The triple integral in the latter equation stands for the so-called influence coefficient, whose derivation is achieved by analytical integration of the Green's function $G_{\zeta\xi}$. Considering the combined influence of tensor components, the Green's function from (11) can be defined as:

$$\begin{cases} G_{ii} = 2\mu\varepsilon_{kii}^* = \mu(u_{ki,i}^* + u_{ki,i}^*); \\ G_{ij} = 2\mu\left(\varepsilon_{kij}^* + \varepsilon_{kji}^*\right) = 2\mu(u_{ki,j}^* + u_{kj,i}^*), \end{cases} \quad (12)$$

where a comma denotes the derivative and $i, j, k = 1,2,3$. If d_{kii} and d_{kij} are the indefinite integrals of the Green's functions (12):

$$d_{122}(x_1,x_2,x_3) = \frac{1}{2\pi}\left(-x_2 \ln(x_3 + r) + (1-2\nu)x_2 \left(\frac{x_3}{2(x_3+r)} + 0.5\ln(x_3+r) \right) \right); \quad (13)$$

$$d_{111}(x_1,x_2,x_3)=\frac{1}{2\pi}\begin{pmatrix}2(1-\nu)x_3\ln(x_2+r)+(1.5-\nu)x_2\ln(x_3+r)+x_1\tan^{-1}\left(\frac{x_2x_3}{x_1r}\right)+...\\ +4(1-\nu)x_1\tan^{-1}\left(\frac{x_2+x_3+r}{x_1}\right)-(1-2\nu)\frac{x_2x_3}{2(x_3+r)}\end{pmatrix}; \tag{14}$$

$$d_{133}(x_1,x_2,x_3)=\frac{1}{2\pi}\left(-2\nu x_3\ln(x_2+r)+(1-2\nu)\left(2x_1\tan^{-1}\left(\frac{x_2+x_3+r}{x_1}\right)+x_2\ln(x_3+r)\right)\right); \tag{15}$$

$$d_{112}(x_1,x_2,x_3)=\frac{1}{\pi}\left(x_3\ln(x_1+r)+2x_2\tan^{-1}\left(\frac{x_1+x_3+r}{x_2}\right)+(0.5-\nu)\left(x_1\ln(x_3+r)+\frac{x_1x_3}{x_3+r}\right)\right); \tag{16}$$

$$d_{113}(x_1,x_2,x_3)=\frac{1}{\pi}\left(x_2\ln(x_1+r)+2x_3\tan^{-1}\left(\frac{x_1+x_2+r}{x_3}\right)\right); \tag{17}$$

$$d_{123}(x_1,x_2,x_3)=-r/\pi; \tag{18}$$

$$d_{2ij}(x_1,x_2,x_3)=d_{1ij}(x_2,x_1,x_3),\ i,j=1,2,3;; \tag{19}$$

$$d_{311}(x_1,x_2,x_3)=\frac{1}{2\pi}\left((1-2\nu)x_3\left(\tan^{-1}\left(\frac{x_2x_3}{x_1r}\right)-\tan^{-1}\left(\frac{x_2}{x_1}\right)\right)-2\nu x_1\ln(x_2+r)\right); \tag{20}$$

$$d_{322}(x_1,x_2,x_3)=d_{311}(x_2,x_1,x_3); \tag{21}$$

$$d_{333}(x_1,x_2,x_3)=\frac{1}{2\pi}\left(2(1-\nu)\left(x_3\tan^{-1}\left(\frac{x_1}{x_3}\right)+x_2\ln(x_1+r)+x_1\ln(x_2+r)\right)+(2\nu-1)x_3\tan^{-1}\left(\frac{x_1x_2}{x_3r}\right)\right); \tag{22}$$

$$d_{312}(x_1,x_2,x_3)=\frac{2}{\pi}\left(-2\nu r-(1-2\nu)x_3\ln\left(x_3+r\right)\right); \tag{23}$$

$$d_{323}(x_1,x_2,x_3)=\frac{1}{\pi}\left(x_1\tanh^{-1}\left(\frac{r}{x_3}\right)-x_2\tan^{-1}\left(\frac{x_1x_3}{x_2r}\right)\right); \tag{24}$$

$$d_{313}(x_1,x_2,x_3)=d_{323}(x_2,x_1,x_3)\text{, with } r=\sqrt{x_1^2+x_2^2+x_3^2}, \tag{25}$$

the influence coefficients $D_{\ell\zeta\xi}$ can be calculated with:

$$D_{\ell\zeta\xi}(i,j,k)=\begin{pmatrix}d_{\ell\zeta\xi}(x_1(i)+\frac{\Delta_1}{2},x_2(j)+\frac{\Delta_2}{2},x_3(k)+\frac{\Delta_3}{2})+d_{\ell\zeta\xi}(x_1(i)+\frac{\Delta_1}{2},x_2(j)-\frac{\Delta_2}{2},x_3(k)-\frac{\Delta_3}{2})+\\ d_{\ell\zeta\xi}(x_1(i)-\frac{\Delta_1}{2},x_2(j)+\frac{\Delta_2}{2},x_3(k)-\frac{\Delta_3}{2})+d_{\ell\zeta\xi}(x_1(i)-\frac{\Delta_1}{2},x_2(j)-\frac{\Delta_2}{2},x_3(k)+\frac{\Delta_3}{2})-\\ d_{\ell\zeta\xi}(x_1(i)+\frac{\Delta_1}{2},x_2(j)+\frac{\Delta_2}{2},x_3(k)-\frac{\Delta_3}{2})-d_{\ell\zeta\xi}(x_1(i)+\frac{\Delta_1}{2},x_2(j)-\frac{\Delta_2}{2},x_3(k)+\frac{\Delta_3}{2})-\\ d_{\ell\zeta\xi}(x_1(i)-\frac{\Delta_1}{2},x_2(j)+\frac{\Delta_2}{2},x_3(k)+\frac{\Delta_3}{2})-d_{\ell\zeta\xi}(x_1(i)-\frac{\Delta_1}{2},x_2(j)-\frac{\Delta_2}{2},x_3(k)-\frac{\Delta_3}{2})\end{pmatrix}, \tag{26}$$

leading to the following computational form of (11):

$$u_k^r(i,j,0)=\sum_{(\ell,m,n)\in\Omega_p}\varepsilon_{\zeta\xi}^p(\ell,m,n)D_{k\zeta\xi}\left(\ell-i,m-j,n\right). \tag{27}$$

The interchanging of the integration and of the observation points discussed above leads to the final form

$$u_k^r(i,j,0)=\sum_{(\ell,m,n)\in\Omega_p}D_{k\zeta\xi}\left(i-\ell,j-m,n\right)\varepsilon_{\zeta\xi}^p(\ell,m,n), \tag{28}$$

which is a discrete convolution product that can be efficiently calculated with DCFFT [16,17].

4. CALCULATION EXAMPLES

A cube of side lengths $2a$, with a fixed but otherwise arbitrarily chosen, containing uniform plastic strains, was considered near the border of an elastic half-space. The cube center is placed at a depth $h=10a$ with respect to the half-space boundary. A Matlab computer program based on the theoretical framework reviewed in this paper and implementing the newly advanced numerical approach was used to predict the residual displacement of the points in the half-space limiting plane. To this end, a Cartesian coordinate system was placed with its origin on the normal to the half-space boundary containing the cube center, and with its x_1 and x_2-axes contained in the limiting plane. The x_3-axis points towards the cube of plastic strains. For the latter, all plastic strains are assumed to vanish except for the plastic strain tensor components specified in table 1, leading to five different configurations of plastic strains. For each case, i.e. for each configuration, dimensionless residual strains in the limiting plane (i.e., in the plane $x_3=0$) are computed with the proposed computer program and the results are depicted in figures 1-4. These predicted distributions agree well with data presented in the literature [18] and prove the robustness of the implemented numerical tool.

Table 1. Configurations of elementary plastic strains

Configuration (case) index	Non-vanishing plastic strain tensor components
1	$\varepsilon_{11}^p=-\varepsilon_{22}^p=0.001$
2	$\varepsilon_{11}^p=-\varepsilon_{33}^p=0.001$
3	$\varepsilon_{12}^p=0.001$
4	$\varepsilon_{13}^p=0.001$
5	$\varepsilon_{23}^p=0.001$

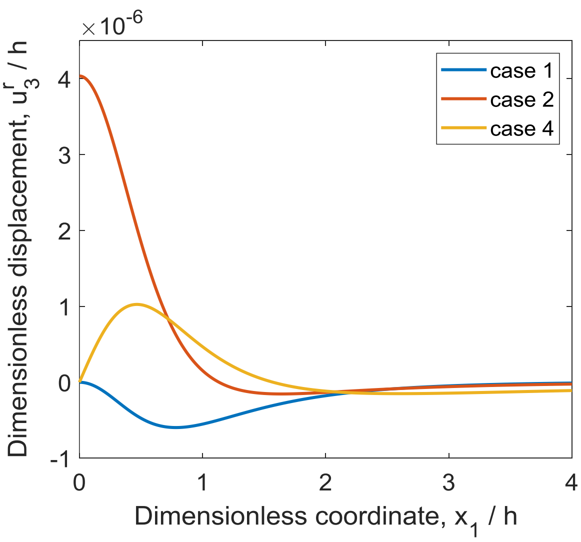

Fig. 1. Dimensionless residual displacement u_3^r along the x_1-axis

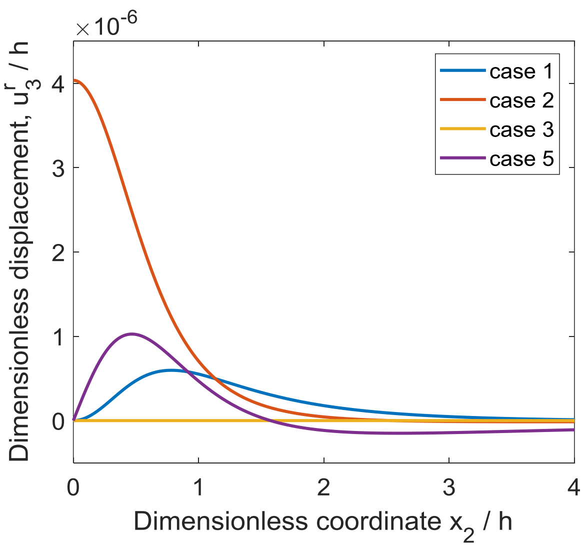

Fig. 2. Dimensionless residual displacement u_3^r along the x_2-axis

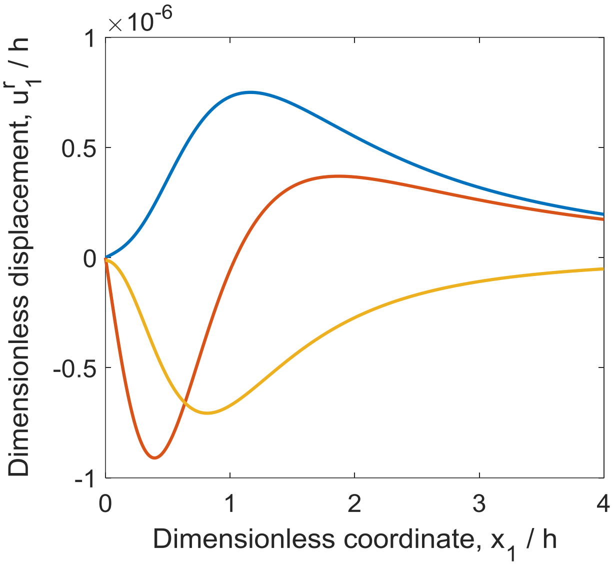

Fig. 3. Dimensionless residual displacement u_1^r along the x_1-axis

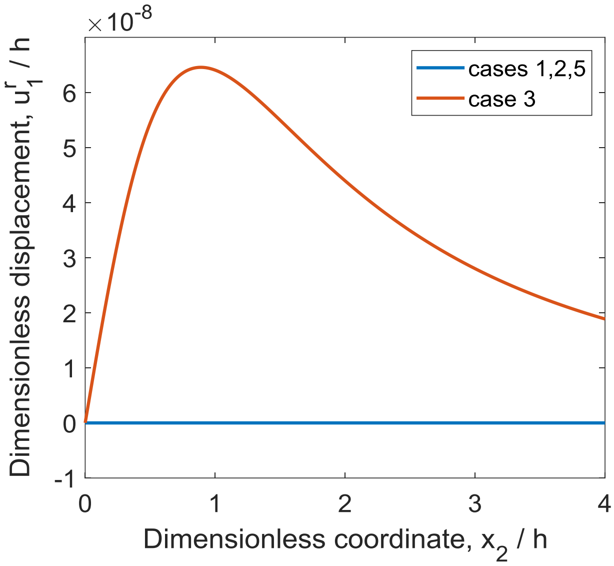

Fig. 4. Dimensionless residual displacement u_1^r along the x_2-axis

5. CONCLUSIONS

A method for the computation of the residual displacement persisting in bodies with initial arbitrary, yet known, plastic strains is advanced and verified against results from the literature.

The theoretical background for the application of superposition of effects to plasticity is established through the Betti's reciprocal theorem particularised to plastic strains. The displacement equation results as a convolution product for which the appropriate Green's functions are defined.

The continuous mathematical model is then discretised to circumvent integration, which is substituted by multi-summation of individual contributions of elementary cuboids containing uniform plastic strains. Indefinite integrals of the Green's functions are derived, leading to the influence coefficients needed to compute the arising convolution product.

The advanced computational model is implemented in a Matlab program that is further used to predict the residual displacement due to specific configurations of plastic strains considered in the literature, and a good agreement is found. The advanced computational model is expected to further the solution of the elastic-plastic contact problem in which the computation of displacement is a critical step.

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