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THERMOELASTIC DISPLACEMENT AND TEMPERATURE RISE IN A HALF-SPACE DUE TO A STEADY-STATE HEAT FLUX

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Abstract: Due to model complexity, classical contact mechanics theory assumes isothermal contact processes, involving bodies with uniform temperatures and no heat transmitted or generated through or near the contact interface. This paper addresses the problem of frictional heating in non-conforming or rough contacts by investigating the thermoelastic behaviour of asperities. The heat generated in a sliding contact by interfacial friction leads to thermoelastic distortion of the contact surface, further modifying contact parameters such as pressure, gap or temperature. The thermal expansion of the contacting bodies must therefore be accounted for when solving the contact problem. The thermoelastic displacement is computed with the aid of the half-space theory and of fundamental solutions for point sources of heat located at the free surface, derived in the literature of heat conduction in solids. The linearity of conduction equations encourages the use of superposition principle in the same way as for the elastic displacement. As the thermoelastic displacement is expressed mathematically as a convolution product, methods derived in contact mechanics for elastic displacement calculation are adapted to the heat conduction equations. The influence coefficients needed to efficiently compute the convolution products are derived, and the Discrete Convolution Fast Fourier Transform technique is applied to improve the algorithm computational efficiency. A similar method is then advanced for the temperature rise on the contact interface due to arbitrary heat input. The predictions of the newly advanced computer programs are tested against existing closed-form solutions for uniform circular or ring heat sources, and a good agreement is found.
Key words: thermoelastic contact, thermal distortion, frictional heating, fast Fourier transform, convolution.

1. INTRODUCTION

The mechanical failure of tribological components is often caused by frictional heating. Integration of thermal parameters and dependencies into the study of tribological processes responsible for contact failures such as scuffing, cracking or seizure, is thus mandatory for contact simulations. A frictional heating induced by the rubbing of the contacting solids leads to a temperature rise, also referred to as flash temperature, and to distortion (or displacement) of the surfaces in the contact region. This further affects the contact geometry and interface parameters such as the contact area and the pressure distribution. The analysis of the thermoelastic distortion and of its interaction with the other contact parameters is thus a critical step for the realistic investigation of contact performance.
However, due to model complexity, in most flash temperature models [1-3], the pressure derivation is performed independently of the heat conduction phenomena and the heat is considered a result of the latter pressure. Other studies [4-6] directly consider an imposed heat source on the half-space boundary. The complex interaction between contact pressure, flash temperature and elastic displacement in a frictional contact was first considered by these authors [7-9]. In the latter works, the contacting bodies are idealised as elastic half-spaces and the displacement analyses are performed based on fundamental half-space solutions known as the Green's functions, expressing the half-space response to a unit impulse, i.e. a point force [10] or a point heat source [11].
The linear nature of both heat conduction and elasticity equations authorises superposition that translates mathematically to integrals in the form of convolution products. However, when the thermoelastic contact problem is discretised, the calculation of these convolutions becomes very computationally intensive, having an order of computations equal to the square of the number of discrete nodes in the mesh for a two-dimensional grid. This computational load has been a limiting factor in the development of computational contact mechanics until the development of techniques based on the fast Fourier transform (FFT). The pioneering works of Ju and Farris [12,13] set a new trend that led to the development of the so-called semi-analytical methods (SAM) for contact analyses. These

methods have a clear advantage over the finite element analyses (FEA) in terms of computational efficiency: with similar computational resources, SAM produces a three-dimensional contact simulation whereas FEA just a two-dimensional one.

An important breakthrough was achieved with the derivation of the DCFFT technique [14], capable of rapid convolution calculation in the frequency domain. Various paths and methods for the application of spectral methods to convolution calculations were explored and compared in [15]. It is noteworthy that the DCFFT extends the types of problems that can be solved to problems with Green's functions derived in closed-form in the frequency domain only, such as the case of multi-layered materials [16], or to contact scenarios involving linear viscoelastic materials [17]. The DCFFT was successfully applied [18] in the study of transient thermoelastic displacement, providing a convenient substitute to time/space domain solutions whose treatment is affected by singularities. A review of the various applications of the spectral methods in contact mechanics and associated problems can be found in [19].

In this paper, a numerical approach is adopted to circumvent the double integration in the convolution product, leading to a robust method capable of assessing the thermoelastic displacement due to arbitrary, yet known, heat sources. The method is expected to contribute to the solution of the sliding contact problem with frictional heating, and to advance the understanding of thermoelastic distortion in mechanical contacts, thus improving tribological design and preventing contact failure.

2. HEAT CONDUCTION IN CONTACT MECHANICS

This section outlines some of the basic concepts and results that allow integration of heat conduction theory into contact mechanics, whereas a complete description of the framework can be found in [11]. The classical theory of contact mechanics assumes isothermal contact processes, meaning the temperature of the contacting bodies remains uniform while load is transmitted through the contact. This assumption may be too strong for many practical engineering scenarios. As the temperature within the contacting bodies may vary, the contact conditions may be affected by both the thermal stresses developed within the bodies, as well as by the thermal distortion of the surface profiles. For example, in case of a non-conforming contact, i.e., involving bodies with dissimilar surface profiles, if the bodies are maintained at different temperatures, heat flows through the contact area, whereas the gap between the bounding surfaces in non-contacting zones can be treated as an insulator. As a result, near the contact region the profile of the cold body will become more convex, whereas that of the hot body will evolve into a less convex or concave profile. These modifications are expected to change the shape and the size of the contact area, as well as the pressure distribution. One exception is when the expansion of the cold body matches exactly the contraction of the hot body, which occurs in case of materials with similar elastic and thermal properties.

Another instance in which the thermal component plays an important role is when heat is generated at or in the vicinity of the contact region, as in the case of frictional heating in sliding contacts or rolling contacts. A case of practical importance is also the local heating of the contact region between non-conforming bodies due to the passage of heavy electric current through the minute contact area.

The analysis of the elastic stresses in thermal contact processes is usually performed in an uncoupled manner due to the complexity of the interactions. The thermal expansion of the contacting bodies is first assessed based on an analysis of the heat conduction or partition between the solids. The resulting thermal distortion of the surface profiles is then inputted into an isothermal contact problem model aiming to find the contact stresses. This approach may not be best suited to sliding contacts, in which the thermal stresses are not independent of the contact conditions. In the latter case, heat is liberated at the contact interface proportional to the contact pressure. In its turn, the contact pressure depends on the distorted surface profiles, which themselves are driven by the distribution of heat. The contact model developed in this paper takes into account this mutual interaction and therefore is well adapted to the analysis of frictional heating in sliding contacts.

In the theory of contact mechanics, the half-space assumption treats bodies of arbitrary profile as elastic half-spaces bounded by a plane surface, thus allowing a convenient calculation of displacement by superposition of solutions for point forces acting on the half-space boundary, i.e., the Green's functions. A similar assumption is used here for thermal effects. The linearity of conduction equations suggests that superposition can be used in calculation of temperature due to an arbitrary heat input, in the same manner as the elastic displacements. To this end, the Boussinesq and Cerruti solutions [20] for point forces have to be replaced by the solutions for point sources of heat placed on the half-space boundary.

This paper is concerned with the situation in which heat is supplied to the contact region at a (total) steady rate $\dot{H}$, i.e a continuous point source, as opposed to an instantaneous source for which the heat is liberated instantaneously at a specific time moment. This situation is of interest because in a sliding contact, the heat rate per unit area $\dot{h} = \dot{H}/A_C$, with A_C the contact

area, is proportional to the sliding velocity v, to the frictional coefficient μ, assumed uniform over the contact area and constant in time, and to the pressure distribution p.

$$\dot{h} = \mu v p . \quad (1)$$

The Green's function for the steady-state (i.e., reached after a sufficient time has elapsed from the initial heating) temperature rise ΔT attained in the vicinity of the continuous point source $\dot{H}$ can be expressed as [11]:

$$\Delta T(r) = T(r) - T_0 = \dot{H}/(2\pi K r) , \quad (2)$$

where T_0 is the (initially uniform) temperature of the half-space and K its conductivity. Integration in a cartesian coordinate system yields the steady-state temperature rise due to a steady rate of heat supplied per unit area (i.e., heat flux) $\dot{h}$:

$$\Delta T(x_1, x_2) = \frac{1}{2\pi K} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\dot{h}(x_1', x_2') dx_1' dx_2'}{\sqrt{(x_1 - x_1')^2 + (x_2 - x_2')^2}} , \quad (3)$$

The temperature rise due to a circular heat source, having a heat flux $\dot{h}$ uniform within a circle of radius a, results as [21]:

$$\Delta T(r) = \frac{2\dot{h}}{\pi K} \int_0^{\pi/2} \sqrt{a^2 - r^2 \sin^2 \alpha}\, d\alpha , \quad r \le a . \quad (4)$$

When heat is transmitted at a constant rate $\dot{h}$ per unit area to the surface of a half-space, the equation of the half-space distortion, i.e., of the normal displacement u_3 due to the supplied heat can be expressed as [11]:

$$u_3(x_1, x_2) = \frac{C}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dot{h}(x_1', x_2') \ln\left(\frac{\sqrt{(x_1 - x_1')^2 + (x_2 - x_2')^2}}{r_0} \right) dx_1' dx_2' . \quad (8)$$

Some cases of analytical integration [4,20] yield the solution for a total heat rate $\dot{H}$ uniformly distributed along an annulus of radius a:

$$u_3(r) = \begin{cases} C\dot{H}/(2\pi) \ln\left(r_0/a\right), & |r| \le a; \\ C\dot{H}/(2\pi) \ln\left(r_0/|r|\right), & |r| > a, \end{cases} \quad (9)$$

as well as for a uniform circular heat of the same radius:

$$\frac{\partial^2 u_3}{\partial x_1^2} + \frac{\partial^2 u_3}{\partial x_2^2} = C\dot{h}(x_1, x_1) , \quad (5)$$

where C is the distortivity of the material:

$$C = (1+\nu)\theta/K , \quad (6)$$

with ν the Poisson's ratio and θ the thermal expansion coefficient. Equation (5) clearly suggests the relation between the distorted surface curvature and the heat flux: locally, the contacting body becomes more convex when heat is flowing into the contacting surface and more concave when heat is flowing out.
The Green's function for the thermoelastic displacement expresses the normal displacement u_3 of a point on the boundary of the half-space, located at a radial distance r from the origin in which a continuous point source of heat $\dot{H}$ is present:

$$u_3(r) = \frac{C\dot{H}}{2\pi} \ln(r/r_0) , \quad (7)$$

where r_0 is a reference point where it is assumed that $u_3(r_0) = 0$. Practically, relation (7) suggests that only the relative displacement between the observation point and the reference point can be calculated. A similar situation can be found in the contact mechanics when deriving the displacement in two-dimensional contact problem (i.e., the contact between infinitely long cylinders). It follows from (7) that the half-space distortion increases without limit if the reference point is considered at increasing distance from the heat source. Integration of (7) in a cartesian coordinate system leads to the (relative) normal displacement due to a steady heat flux $\dot{h}$:

$$u_3(r) = \begin{cases} C\dot{H}/(4\pi)\left(2\ln(r_0/a) + 1 - r^2/a^2\right), & |r| \le a; \\ C\dot{H}/(2\pi) \ln(r_0/|r|), & |r| > a. \end{cases} \quad (10)$$

The solution of a contact problem for arbitrary contact geometry can only be achieved numerically [20,22] in an iterative manner. In the iterative process, both contact area A_C and pressure distribution p can achieve arbitrary distributions. Consequently, relation (1) suggests that the heat flux $\dot{h}(x_1, x_2)$ can also be

arbitrary. The integrals (3) and (8), which are in fact convolution products between the heat flux and the Green's functions, must be re-evaluated at each iteration. An efficient computational technique for the numerical evaluation of integrals (3) and (8) for arbitrary heat flux is explored in the next section.

3. COMPUTATIONAL TECHNIQUE

Relation (3) is formally analogous to that of the surface displacement due to arbitrary pressure, provided pressure is substituted by the heat flux and the contact compliance by the material conductivity K. This resemblance suggests that similar computational methods can be applied. The displacement computation by numerical analysis is one of the fundamental problems in contact mechanics, as the computational complexity of the convolution products was a limiting factor for the solution of contact problems. Practically, some contact problems that require fine meshes were considered unsolvable due to the prohibitive computational effort. The Multi-Level Multi-Summation was successfully applied [22] to solve the contact problem of rough surfaces. However, the Discrete Convolution Fast Fourier Transform (DCFFT) presents itself as a more robust alternative that is intensively used in computational contact mechanics since its derivation [14].

The DCFFT is well adapted to the current problem, as the needed Green's functions have already been expressed in the time/space domain, i.e. equations (3) and (8). The DCFFT is basically an algorithm for numerical approximation of a continuous linear convolution between two functions G and f:

$$O(t) = \int_{-\infty}^{\infty} G(t-\tau) f(\tau) \equiv G(t) \otimes f(t), \qquad (11)$$

where the "$\otimes$" symbol is used to denote the convolution product operation. Relations (3) and (8) are both of the same type as (11), by substituting G with the (two-dimensional) Green's functions (2) and (7), and f with the heat flux $\dot{h}$. If relation (11) is transferred into the frequency domain (with ω the variable in the frequency domain), the convolution theorem establishes the relation between the transformed functions, marked with a "~" symbol, as a simple product operation:

$$\tilde{O}(\omega) = \tilde{G}(\omega) \cdot \tilde{f}(\omega), \qquad (12)$$

It is clear that from a computational point of view, equation (12) is more convenient than (11). As f can be arbitrary in a contact process solution, convolutions (11) or (12) must be evaluated numerically, thus the convolution members G and f must be digitized on a finite domain. The continuous linear convolution is thus substituted by a cyclic circular convolution between the series resulted from the digitization of the continuous functions:

$$\hat{O}(\omega) = \hat{G}(\omega) \cdot \hat{f}(\omega), \qquad (13)$$

where a different symbol "$\wedge$" was used for discrete series in the frequency domain. If $\hat{G}$ and $\hat{f}$ are series with N terms, the calculation of the convolution product is an operation of order $O(N^2)$ in the time/space domain, whereas relation (13) is of order $O(N)$. A specific method error, referred to as the periodicity error, manifests itself when the discrete relation (13) on a finite domain is used for the approximation of its continuous counterpart (11) on an infinite domain. In the DCFFT, the latter error is completely removed by performing the element-wise product (13) on a domain twice (in every dimension) the target domain, i.e. the domain for which the results are needed. The main steps of the DCFFT algorithm are outlined below, whereas full details concerning the algorithm implementation can be found elsewhere [23].

1. Establish the computational domain and digitize the convolution members on it, thus obtaining series in the time/space domain.
2. Apply re-arrangements in the series to circumvent the periodicity error (zero-padding and wrap-around order).
3. Transfer the series in the frequency domain by fast Fourier transform algorithm (FFT), thus obtaining spectral series of the same size as the original ones.
4. Perform the element-wise product of these series according to equation (13). At this point, the convolution result is obtained in the frequency domain.
5. Apply inverse FFT to the series calculated in the previous step and truncate to resulting series to the target domain.

As the transfers to and from the frequency domain by FFT and inverse FFT both have a computational order of $O(N \log N)$, it follows that the DCFFT is of order $O(N \log N)$, which is an important reduction compared to the number of operations required for the convolution calculation in the time/space domain, i.e. $O(N^2)$. The reduction becomes of critical importance when $N > 10^3$, which is considered as a minimum resolution for rough contact problems.

Digitization of the convolution member that stands for the heat input, i.e., f, is straightforward and involves the assumption of a piece-wise constant distribution. A

fixed-step (on every direction, but can differ from one direction to another) rectangular grid is used and the heat flux $\dot{h}$ is assumed uniform on every rectangular cell, but can vary from one cell to another. If a continuous function is given, digitization is performed based on a chosen representative point, e.g., the center of each rectangular cell. In contact problems, f is the distribution that needs to be computed iteratively, so an initial approximation is required, usually taking the form of a mean value over the target domain.

In order to reduce the discretisation error, digitization of the Green's function G is performed by using the concept of influence coefficients, which are integrals of the Green's function over a domain matching the elementary cell of the imposed mesh. For the convolutions (3) and (8), if a mesh step size of $2\Delta_i$ is assumed along direction of $\vec{x}_i$, with $i=1,2$, the influence coefficients can be computed analytically from the function $D(x_1,x_2)$:

$$D(x_1,x_2)=\int_{-\Delta_2}^{\Delta_2}\int_{-\Delta_1}^{\Delta_1} G(x_1-x_1',x_2-x_2')dx_1'dx_2'= \\ =g(x_{1u},x_{2u})+g(x_{1\ell},x_{2\ell})-g(x_{1u},x_{2\ell})-g(x_{1\ell},x_{2u}) \\ , \tag{14}$$

where g is the indefinite double integral of G, and $x_{iu}=x_i+\Delta_i$, $x_{i\ell}=x_i-\Delta_i$, $i=1,2$. The influence coefficients series (matrix) results from (14) by substituting the parameters (x_1,x_2) with all possible distances between all representative points of the cells in the mesh. E.g., for a linear mesh with N elementary domains there exist $2N$ discrete influence coefficients, as the relative position of the points relative to origin must also be considered. $D(x_1,x_2)$ thus describes the interaction between any source point and any observation point matching the conditions that $x_i=x_{io}-x_{is}$, $i=1,2$, with x_{io} the coordinates of the observation (computation) point and x_{is} those of the heat source point. The convolution products of type (11) are needed because, in the superposition of effects, every point in the mesh is an observation as well as a source point. The double integrals for the calculation of the influence coefficients corresponding to the Green's functions (2) and (7) are given below:

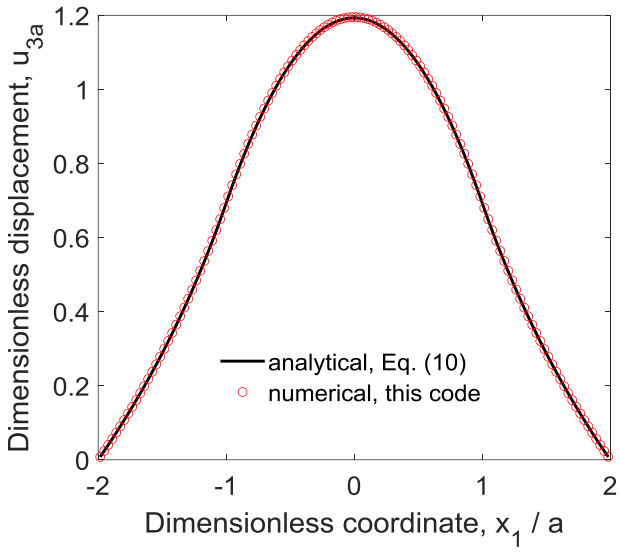
$$g^{\Delta T}(x_1,x_2)=\frac{1}{2\pi K}\left(x_1\ln\left(x_2+\sqrt{x_1^2+x_2^2}\right)+x_2\ln\left(x_1+\sqrt{x_1^2+x_2^2}\right)\right), \tag{15}$$

$$g^{u_3}(x_1,x_2)=\frac{C}{2\pi}\left(x_1x_2\ln\left(\frac{\sqrt{x_1^2+x_2^2}}{r_0}\right)+\frac{1}{2}\left(x_1^2\tan^{-1}\left(\frac{x_2}{x_1}\right)+x_2^2\tan^{-1}\left(\frac{x_1}{x_2}\right)\right)-\frac{3x_1x_2}{2}\right). \tag{16}$$

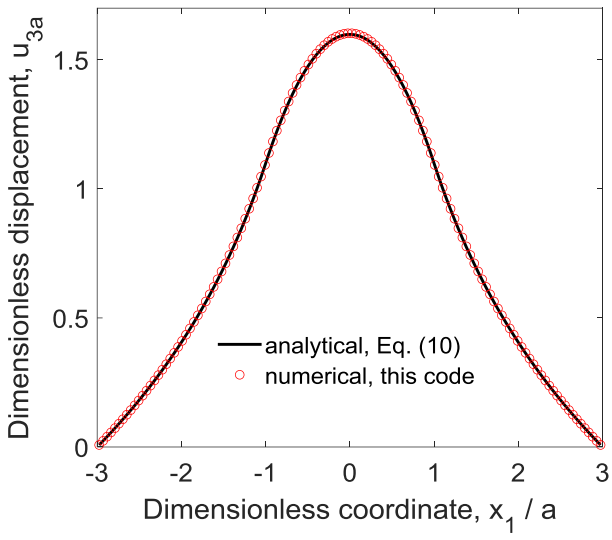
4. MODEL VALIDATION

The newly proposed computational technique is benchmarked against exiting solutions derived in the literature [4,20]. A steady heat flux $\dot{h}$ uniformly distributed within a circle of radius a, with a total rate $\dot{H}=\dot{h}\pi a^2$, induces on the boundary of a half-space a temperature rise given by relation (4) that is not expressed in closed-form but can be easily integrated numerically, and a thermoelastic displacement (10) depending upon a datum r_0. These quantities are computed here with the DCFFT-based approach proposed in the previous section, and the resulting distributions are compared in figures 1 and 2 for different a/r_0 ratios. Dimensionless displacement u_{3a} is defined as $u_{3a}=2\pi u_3/(C\dot{H})$.

In all cases, a good agreement is found. The depicted distributions clearly show that vanishing displacement, i.e., $u_3=0$, is attained at a radial distance $r=r_0$.



(a)



(b)

Fig. 1. Thermoelastic distortion due to a circular heat source: a) $r_0 = 2a$; b) $r_0 = 3a$.

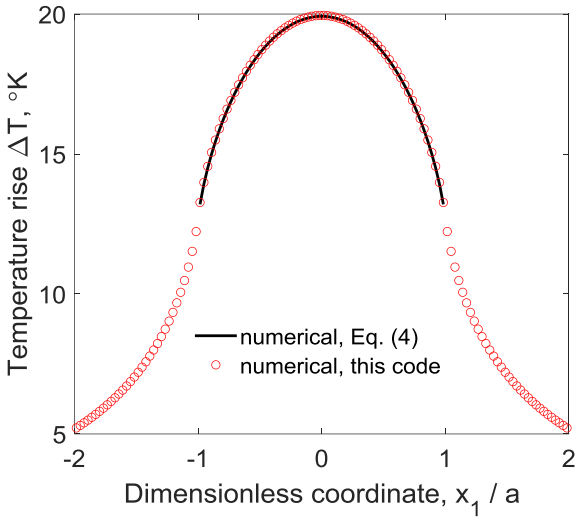


Fig. 2. Temperature rise due to a circular heat source, $\dot{H} = 0.2\pi\mathrm{W}$, $K = 50.2\,\mathrm{W/(m°K)}$, $a = 0.2\mathrm{mm}$.

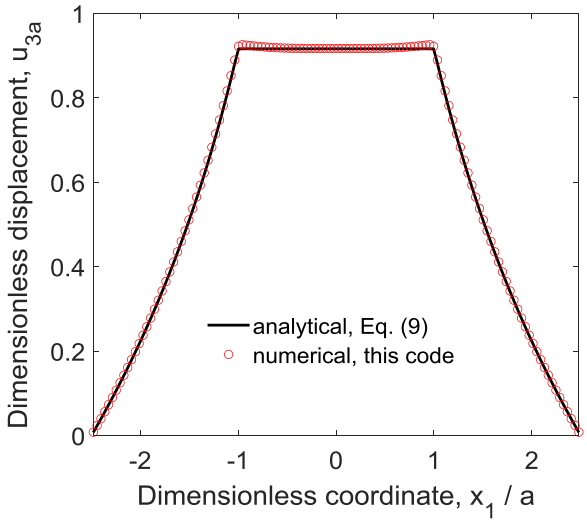


Fig. 3. Thermoelastic distortion due to a ring heat source.

An additional verification considers the case of the ring heat source whose thermoelastic displacement is given by equation (9). It should be noted that in the latter case, the analytical result inside the heat annulus is identical to that for a point source, which is a mathematical idealization as heat can only be transmitted through a finite area. As a circle also has vanishing area, the closed-form expression (9) uses the total heat rate $\dot{H}$ instead of the heat flux $\dot{h}$. In the numerical calculation, a set of elementary rectangular patches was selected to approximate a circle of a given radius in the discrete space. The latter process has no unique solution as a threshold for the selection has to be chosen, which is dependent on the resolution of the imposed grid. In our simulation, the aforementioned threshold was varied until the number or elementary patches on the circumference was the closest to $2\pi n$, with $n = a/(2\Delta_1)$ and $\Delta_1 = \Delta_2$. The obtained distribution departs slightly from the analytical solution at the ring boundary, $x_1/a = \pm 1$, which is to be expected considering the limited capability of a reunion of rectangles to simulate a circle. Nevertheless, considering the aforementioned limitations, the validation depicted in figure 3 is considered to be satisfactory.

These validations give confidence in the proposed method and computer program and suggests that this framework can be further incorporated into a contact model with thermal boundary conditions expressed also in terms of steady heat flux.

5. CONCLUSIONS

The goal of the paper is to advance a numerical technique allowing the calculation of normal displacement due to an arbitrary steady-state heat flux

liberated on a limited region of a conductive half-space. The result is of great interest for the solution of the sliding contact problem with the consideration of frictional heating influence on the contact parameters. Results from the heat conduction theory are integrated in a formulation employed in computational contact mechanics to derive the displacement and stress fields due to arbitrary contact stresses. The solution for heat source points is employed in the same manner as the solution for point forces acting on the half-space boundary. Application of superposition principle leads to integrals in the form of double convolution products.

A state-of-the-art method for rapid calculation of convolution products in the Fourier transform domain is applied and the needed influence coefficients are derived. The computer program is benchmarked against existing solutions for the displacement induced by circular or ring uniform heat sources. The good agreement gives confidence in the proposed method, although the results depend on a reference point. The same computational module is applied and validated in the calculation of temperature rise due to a circular uniform heat source.

An integration of the proposed numerical technique into a contact model, with the consideration of displacements due to both contact tractions and thermal effects, is intended for the future research.

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